ProSeminar QCD

Yang-Mills Theory and the QCD Lagrangian

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Contents

1 Introduction

Quantum electrodynamics introduce the important feature of local gauge invariance. The QED gauge group is the abelian group $U(1)$, which means that the QED Lagrangian is invariant under local phase shifts.
The natural generalization of the $U(1)$ group as a gauge group is the $SU(2)$ group. By looking at it one finds that its generators are not commuting. $SU(2)$ is therefore a non-abelian group. In 1954 Yang and Mills took this group as the gauge group of the isotopic spin, which later lead to the unification of weak and electromagnetic interaction.
We will develop these gauge symmetries first in the case of QED and later on its generalization to the non-abelian case. This generalization will lead us to the Lagrangian of Quantumchromodynamics[1] and the introduction of Quarks and Gluons, which are the two participants of QCD.

2 Gauge Symmetries

In this section we will shortly review the principle of local gauge invariance in the example of QED and afterwards give a slightly longer development of non-abelian gauge symmetry. This will lead us to the Yang-Mills Lagrangian in the end.

2.1 Abelian Gauge Symmetry

In QED one sees that the Lagrangian is invariant under local[2] phase rotation symmetry. Nowadays however this invariance is not anymore seen as a result of the theory, but as an assumption, which determines the theory. Let us begin by assuming the invariance of the theory under the following transformation:

$$\psi(x) \longrightarrow e^{i\alpha(x)}\psi(x) \tag{2.1}$$

which, up to first order in α, is

$$\psi(x) \longrightarrow \psi(x) + i\alpha(x)\psi(x). \tag{2.2}$$

Since our aim is to construct a Lagrangian that is invariant under this transformation, we must also glance at derivatives of $\psi(x)$ since in general a Lagrangian is a function

$$\mathcal{L} = \mathcal{L}(\psi(x), \partial\psi(x), t).$$

This is where the to interesting results leading difficulties arise. The difficulty is that at different points in spacetime the derivative of $\psi(x)$ in direction n^μ

$$n^\mu \partial_\mu \psi = \lim_{\epsilon \to 0} \frac{\psi(x + \epsilon n) - \psi(x)}{\epsilon} \tag{2.3}$$

[1] henceforth called QCD
[2] i.e. x-dependent

transforms differently under the local transformation (2.1). We therefore define a scalar quantity $U(y,x)$, called comparator, with the following transformation law:

$$U(y,x) \longrightarrow e^{i\alpha(y)}U(y,x)e^{-i\alpha(x)} \tag{2.4}$$

and we can require it to be a pure phase $U(y,x) = e^{i\varphi(y,x)}$. As one easily can verify $\psi(y)$ and $U(y,x)\psi(x)$ have the same transformation law, so we can compare these fields at different points. This gives meaning to the subtraction in (2.3) and we define the *covariant derivative*

$$n^\mu D_\mu \psi = \lim_{\epsilon \to 0} \frac{\psi(x+\epsilon n) - U(x+\epsilon n, x)\psi(x)}{\epsilon}. \tag{2.5}$$

Since the comparator can be seen as a pure phase, we can expand it

$$U(x+\epsilon n, x) = \underbrace{U(x,x)}_{:=1} - ie\epsilon n^\mu A_\mu(x) + \mathcal{O}(\epsilon^2) \tag{2.6}$$

where e is an arbitrary constant. Plugging this into the definition of the covariant derivative we get

$$\begin{aligned} n^\mu D_\mu \psi &= \lim_{\epsilon \to 0} \frac{\psi(x+\epsilon n) - (1 - ie\epsilon n^\mu A_\mu(x))\psi(x)}{\epsilon} \\ &= \underbrace{\lim_{\epsilon \to 0} \frac{1}{\epsilon}(\psi(x+\epsilon n) - \psi(x))}_{\partial_\mu \psi(x)} + ien^\mu A_\mu(x)\psi(x) \end{aligned} \tag{2.7}$$

$$\Rightarrow D_\mu = \partial_\mu + ieA_\mu(x). \tag{2.8}$$

Given the transformation law of the comparator (2.4) we plug in the expansion (2.6) and find the transformation law of $A_\mu(x)$ to be

$$A_\mu(x) \longrightarrow A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x). \tag{2.9}$$

Now we have all the ingredients to check that $D_\mu \psi(x)$ transforms as the field $\psi(x)$ itself:

$$D_\mu \psi(x) = (\partial_\mu + ieA_\mu(x))\psi(x) \rightarrow \left(\partial_\mu + ie(A_\mu(x) - \frac{1}{\epsilon}\partial_\mu \alpha)\right) e^{i\alpha(x)}\psi(x) \tag{2.10}$$

$$= e^{i\alpha(x)}(\partial_\mu + ieA_\mu(x))\psi(x) \tag{2.11}$$

$$= e^{i\alpha(x)} D_\mu \psi(x). \tag{2.12}$$

Since the Lagrangian as we have seen contains an additional vector field $A_\mu(x)$, we need to have a term containing $A_\mu(x)$ itself as well as its derivative, but which is independent of $\psi(x)$. Such a term is called kinetic energy term of the gauge field.
To find such a term we choose a geometrical derivation where we consider an infinitesimal square and expand the comparator to third order.

Since $U(y,x) = (U(x,y))^\dagger$ and it is a pure phase we get from the expansion (2.6)

$$U(x+\epsilon n, x) = \exp\left(-ie\epsilon n^\mu A_\mu(x+\tfrac{\epsilon}{2}n) + \mathcal{O}(\epsilon^3)\right). \tag{2.13}$$

If we now consider a square and look at the comparator as we walk around this square we can define the locally invariant

$$\mathbf{U}(x) := U(x, x+\epsilon\hat{2})U(x+\epsilon\hat{2}, x+\epsilon\hat{1}+\epsilon\hat{2})U(x+\epsilon\hat{1}+\epsilon\hat{2}, x+\epsilon\hat{1})U(x+\epsilon\hat{1}, x). \tag{2.14}$$

Since $U(x+\epsilon n, x)$ depends only on $A_\mu(x)$ so does $\mathbf{U}(x)$. If we insert (2.13) in the definition of $\mathbf{U}(x)$ we get an exponential function of $A_\mu(x)$, which can be expanded in powers of ϵ:

$$\mathbf{U}(x) = 1 - i\epsilon^2 e \underbrace{\left(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\right)}_{:=F_{\mu\nu}} + \mathcal{O}(\epsilon^3) \tag{2.15}$$

and since $\mathbf{U}(x)$ is locally invariant so is $F_{\mu\nu}$, which may be recognized as the antisymmetric field-strength tensor of the electromagnetic field. More general terms like a mass term $A_\mu A^\mu$ do not transform properly so they cannot be included in the Lagrangian.
Writing explicit expressions for $A_\mu(x)$ and ∂_μ as

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right)$$
$$A_\mu(x) = \left(-\frac{\phi}{c}, \vec{A}\right)$$

we see that the field strength tensor[3] is given as

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{2.16}$$

Since we assume to live in a spacetime of dimension 4 the most general Lagrangian for QED can only contain the field $\psi(x)$ as well as its covariant derivatives and the tensor $F_{\mu\nu}$ as well as its derivatives since it has to be invariant to local and global phase transformations[4]. Additionally the Lagrangian is required to represent a renormalizable theory[5].
The components of the Lagrangian are the fermionic mass term $m\bar{\psi}\psi$, the kinetic term $\bar{\psi}(i\not{D})\psi$ and the electromagnetic field tensor:

$$\mathcal{L} = \bar{\psi}(i\not{D})\psi - m\bar{\psi}\psi - \tfrac{1}{4}(F_{\mu\nu})^2 \tag{2.17}$$

where $\not{D} = \gamma^\mu D_\mu$[6]. The coefficients arise from the normalization of the fields. If we use the above expression for $F_{\mu\nu}$ to calculate the corresponding term in the Lagrangian we get

[3] in natural units to keep it simple
[4] invariance to local phase transformations imply invariance to global ones
[5] Details of this problem will be presented in next weeks talk
[6] see also appendix B

$$-\tfrac{1}{4}(F_{\mu\nu})^2 = \tfrac{1}{2}\left(E^2 - B^2\right) \tag{2.18}$$

which is the familiar energy of the electromagnetic field.
For later use we calculate the commutator of two covariant derivatives:

$$[D_\mu, D_\nu]\psi = ie(\partial_\mu A_\nu - \partial_\nu A_\mu)\psi = ieF_{\mu\nu}\psi. \tag{2.19}$$

2.2 Non-abelian Gauge Symmetry

2.2.1 $SU(2)$

In 1954 Yang and Mills considered a Lagrangian with a non-abelian transformation group in the context of the isospin-doublet and found a general gauge invariant Lagrangian for non-abelian gauge theories. Therefore this Lagrangian is called *Yang-Mills-Lagrangian.*
To generalize the previous results we consider the group $SU(2)$ instead of a phase transformation and instead of a single field the doublet of fields

$$\psi = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}. \tag{2.20}$$

We denote the local $SU(2)$ transformations by

$$\psi \longrightarrow \underbrace{\exp\left(i\alpha^j(x)\frac{\sigma^j}{2}\right)}_{:=V(x)}\psi \tag{2.21}$$

where the σ^j are the Pauli matrices. The difference to the previous section is obvious: the generators of the transformation group do not commute and henceforth form a non-abelian group.
To construct a $SU(2)$-transformation invariant Lagrangian we proceed as in the previous section with the needed adjustments. We first need to find a comparator, which obviously now has to be a 2×2-matrix, due to the doublet of fields.
We define this comparator by its transformation law

$$U(y,x) \longrightarrow V(y)U(y,x)V^\dagger(x) \tag{2.22}$$

and the requirements that $U(y,x)$ needs to be unitary[7] and $U(x,x) = 1$. Again we can expand $U(y,x)$ near the identity[8]

$$U(x+\epsilon n, x) = \underbrace{U(x,x)}_{:=1} + ig\epsilon n^\mu A^i_\mu \frac{\sigma^i}{2} + \mathcal{O}(\epsilon^2) \tag{2.23}$$

[7] due to the famous theorem by Wigner which states that every symmetry transformation can be represented by an unitary and linear operator acting on states in the physical Hilbert space $\ni \psi(x)$

[8] since we are assuming a simply connected space

where g is for the time being an arbitrary constant[9]. Plugging this into the general expression for the covariant derivative (2.5) and going through trivial calculations we find the covariant derivative in this context to be of the form

$$D_\mu = \partial_\mu - igA^i_\mu \frac{\sigma^i}{2}. \tag{2.24}$$

To find the transformation law of A^i_μ we need to insert the expansion of the comparator (2.22) into its transformation law (2.21) and expand the $\alpha(x+\epsilon n)$-term in $V(x+\epsilon n)$ to first order in ϵ[10]:

$$\begin{aligned}\left(1 + ig\epsilon n^\mu A^i_\mu \frac{\sigma^i}{2}\right) &\longrightarrow V(x+\epsilon n)\left(1 + ig\epsilon n^\mu A^i_\mu \frac{\sigma^i}{2}\right) V^\dagger(x) \\ &= \exp\left(i \underbrace{\alpha^j(x+\epsilon n)}_{\alpha^j(x)+\epsilon n^\nu \partial_\nu \alpha^j(x)} \frac{\sigma^j}{2}\right)\left(1 + ig\epsilon n^\mu A^i_\mu \frac{\sigma^i}{2}\right) \exp\left(-i\alpha^k(x)\frac{\sigma^k}{2}\right) \\ &= (1 + i\alpha^j(x)\frac{\sigma^j}{2} + i\epsilon n^\nu \partial_\nu \alpha^j \frac{\sigma^j}{2})\left(1 + ig\epsilon n^\mu A^i_\mu \frac{\sigma^i}{2}\right)(1 - i\alpha^k(x)\frac{\sigma^k}{2}).\end{aligned} \tag{2.25}$$

Performing this calculation, relabeling some indices we sum over, caring about the non-commutativity of the Pauli-matrices, omitting terms of order ϵ^2 and comparing terms proportional to ϵn^μ we find

$$A^i_\mu \frac{\sigma^i}{2} \longrightarrow A^i_\mu \frac{\sigma^i}{2} + \frac{1}{g}(\partial_\mu \alpha^i)\frac{\sigma^i}{2} + i\left[\alpha^i \frac{\sigma^i}{2}, A^j_\mu \frac{\sigma^j}{2}\right] + \dots \tag{2.26}$$

If we compare this result to (2.9) we see that we obtained a new term including the commutator of the non-commutative generators of $SU(2)$.
Inserting this result into our definition of the covariant derivative (2.23) we see that it indeed transforms like the field itself:

$$D_\mu \psi \longrightarrow \underbrace{\exp\left(i\alpha^j \frac{\sigma^j}{2}\right)}_{=V(x)} D_\mu \psi. \tag{2.27}$$

To find all the ingredients for an invariant Lagrangian we must, like in the abelian case, find the "kinetic energy terms" of the gauge field A^i_μ but this time we will use a different method to find them:
Since $D_\mu \psi$ is locally covariant, so is the commutator of the derivatives $[D_\mu, D_\nu]$ as one can easily verify using the transformation law (2.27)

$$[D_\mu, D_\nu]\psi \longrightarrow V(x)[D_\mu, D_\nu]\psi. \tag{2.28}$$

Doing the calculation we find

$$\begin{aligned}[D_\mu, D_\nu]\psi &= -ig\left(\partial_\mu A^j_\nu \frac{\sigma^j}{2} - \partial_\nu A^i_\mu \frac{\sigma^i}{2} - ig\left[A^i_\mu \frac{\sigma^i}{2}, A^j_\nu \frac{\sigma^j}{2}\right]\right)\psi \\ &\equiv -igF^i_{\mu\nu}\frac{\sigma^i}{2}\psi.\end{aligned} \tag{2.29}$$

[9]There will be restrictions on this so-called coupling constant in QCD which fix this number
[10]there is a more elegant but less straight forward way in doing, we choose the straight forward

If we compare this to (2.19) we see that a new term appeared as in the transformation law of A^i_μ (2.26) due to their non-commutativity. Using the standard commutation relations of the Pauli-matrices[11] and relabeling summation indices we find

$$\begin{aligned} F^i_{\mu\nu} &= \partial_\mu A^j_\nu - \partial_\nu A^i_\mu - ig\left[A^i_\mu \frac{\sigma^i}{2}, A^j_\nu \frac{\sigma^j}{2}\right]\frac{2}{\sigma^i} \\ &= \partial_\mu A^j_\nu - \partial_\nu A^i_\mu + g\epsilon^{ijk} A^j_\mu A^k_\nu. \end{aligned} \tag{2.30}$$

2.2.2 General case

In the general case one has to consider general states of the physical Hilbert space, i.e. n-dimensional fields $\psi(x)$. They transform under a continuous group of transformations according to

$$\psi(x) \longrightarrow V(x)\psi(x). \tag{2.31}$$

In group theory one can show that for any continuous group[12] there exists a isomorphism between the corresponding Lie algebra and some subalgebra of $\mathfrak{gl}_n(\mathbb{C})$[13] the elements of the symmetry-transformation group can be expressed as $n \times n$ matrices in the vicinity of the identity.

Since the space of transformations is continuous and simply connected, we can expand its elements around the identity

$$V(x) = 1 + i\alpha^a(x)t^a + \mathcal{O}(\alpha^2). \tag{2.32}$$

Since a continuous group which is also a smooth manifold by definition is a Lie group, the tangential space at the identity defined by the derivation of the group element at the identity[14] is a Lie algebra spanned by the generators t^a. They can be represented by Hermitian matrices[15].

In every algebra there is by definition a multiplication law defined which states that the product of two elements of the algebra must itself lie in the algebra, i.e. it must in the most general case be a linear combination of elements of the algebra. In Lie algebras the multiplication law is the commutator[16] and the linear combination of elements of the group is represented by the so called structure constants f^{ab}_c via

$$[t^a, t^b] = if^{ab}_c t^c. \tag{2.33}$$

For instance in the case of $SU(2)$ the Pauli matrices are the generators of the fundamental representation and its structure constants are the elements of the totally antisymmetric

[11] $\left[\frac{\sigma^i}{2}, \frac{\sigma^j}{2}\right] = i\epsilon^{ijk}\frac{\sigma^k}{2}$

[12] i.e. Lie group

[13] Ado's theorem

[14] $\left.\frac{\partial V(x)}{\partial x}\right|_{x=0}$

[15] t^a must be Hermitian for $V(x)$ to be unitary and linear as one can easily verify

[16] the commutator of elements of tangential spaces, i.e. of elements of vector spaces has to fulfill the Leibnitz rule

tensor ϵ^{ijk}. In general f_c^{ab} is by virtue of the Jacobi-identity[17] antisymmetric in the upper two indices.
The general covariant derivative is always given by

$$D_\mu = \partial_\mu - igA^a_\mu t^a \tag{2.34}$$

as can easily be shown by requiring that $D_\mu\psi$ transforms under the transformation (2.31) as ψ itself.
The infinitesimal transformation law for A^a_μ is given by

$$A^a_\mu \longrightarrow A^a_\mu + \frac{1}{g}\partial_\mu \alpha^a + f^a_{bc}A^b_\mu \alpha^c \tag{2.35}$$

and if one inserts the expansion (2.32) into the transformation law of the comparator, one finds, as in the case of $V(x) \in SU(2)$, the finite transformation of A^a_μ to be

$$A^a_\mu(x)t^a \longrightarrow V(x)\left(A^a_\mu(x)t^a + \frac{i}{g}\partial_\mu\right)V^\dagger(x). \tag{2.36}$$

The field-strength tensor is given by

$$[D_\mu, D_\nu]\psi = ig\underbrace{\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu\right)}_{F^a_{\mu\nu}} t^a\psi \tag{2.37}$$

$$= -igF^a_{\mu\nu}t^a\psi. \tag{2.38}$$

The transformation law of the field-strength tensor is

$$F^a_{\mu\nu}t^a \longrightarrow V(x)F^b_{\mu\nu}t^b V^\dagger(x) \tag{2.39}$$

$$= F^a_{\mu\nu}t^a - f^{abc}\alpha^c F^b_{\mu\nu}t^a. \tag{2.40}$$

We notice that unlike in the abelian case the field-strength tensor is no longer gauge invariant, which is reasonable, since its definition contains terms that reflect the structure of the algebra. In the same manner the expansion of $V(x)$ contains the generators of the algebra, so their product has to include factors that reproduce the structure of the algebra. Another reason can be seen in the dimensionality of the algebra: in the abelian case we only had one one-dimensional generator of the symmetry group, now for a $SU(N)$ algebra we have $N^2 - 1$ generators, and of course the field-strength tensor in general is not the same for each pair of generators.
However, it is quite simple to form gauge invariant combinations of the field-strength tensor, for instance

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{2}\operatorname{Tr}\left[(F_{\mu\nu})^2\right] = -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 \tag{2.41}$$

[17] $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$

is invariant as one can easily verify:

$$-\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 \longrightarrow \mathrm{Tr}[V(x)]\left(-\frac{1}{4}\left(F^a_{\mu\nu}\right)^2\right)\mathrm{Tr}[V^\dagger(x)] \tag{2.42}$$

$$= -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 + \underbrace{\frac{i}{4}\left(F^a_{\mu\nu}\right)^2\alpha^b t^b - \frac{i}{4}\alpha^b t^b\left(F^a_{\mu\nu}\right)^2}_{=\left[\left(F^a_{\mu\nu}\right)^2,\alpha^b t^b\right]=0} - \underbrace{\frac{1}{4}\alpha^b t^b\left(F^a_{\mu\nu}\right)^2\alpha^b t^b}_{\mathcal{O}(\alpha^2)\to 0} \tag{2.43}$$

$$= -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2. \tag{2.44}$$

Our invariant Lagrangian needs to have a mass-term for the fermion field $m\bar{\psi}\psi$, a kinetic term for the fermion field $\bar{\psi}(i\not{D})\psi$ and a term for the gauge field containing the gauge field tensor (2.41):

$$\mathcal{L} = \bar{\psi}(i\not{D})\psi - \frac{1}{4}\left(F^a_{\mu\nu}\right)^2 - m\bar{\psi}\psi. \tag{2.45}$$

This is the most general renormalizable Lagrangian that conserves P and T[18], i.e. is invariant under space and time inversions. Also this Lagrangian contains terms that are quadratic or of even higher powers in the gauge fields A^a_μ. These terms appear because of the non-vanishing commutator in $F^a_{\mu\nu}$ and they represent selfinteractions of the gauge field particles. They appear because unlike in the abelian case these particles, called gauge bosons, on the one hand themselves carry "'charges"' and on the other hand interact with anything that carry "'charge"' so it is almost naturally sensible that they should interact with themselves.
An interesting consequence of this is the theoretical existence of particles consisting only of gluons, so called "'glueballs"'. These glueballs should exist at energy ranges which are already reached with modern colliders, but the identification is quite complicated.
In an abelian gauge theory, specifically in the in section 2.1 discussed $U(1)$ theory, the gauge particles, called photons, carry no "'charge"' of the force they mediate which is the electromagnetic force and hence don't interact with each other.

2.2.3 General remarks on the construction of a Lagrangian

The transformation rules for the field-strength tensor $F^a_{\mu\nu}$ (2.37) as well as the one for the matter field ψ (2.31) and for the covariant derivative $D_\mu\psi$ (2.24) do not involve any terms proportional to derivatives of $\alpha^a(x)$. So if we would construct our general Lagrangian only from these three ingredients and terms of higher order of them it would be guaranteed to be gauge invariant. Generally spoken we could write

$$\mathcal{L} = \mathcal{L}\left(\psi, D_\mu\psi, D_\mu D_\nu\psi, \ldots, F^a_{\mu\nu}, D_\sigma F^a_{\mu\nu}, D_\sigma D_\rho F^a_{\mu\nu}, \ldots\right) \tag{2.46}$$

and be sure that it is gauge invariant.
Also we could give our invariance condition a mathematical form:

[18] which are only asymptotically conserved in weak interactions as was shown in experiments

$$\frac{\partial \mathcal{L}}{\partial \psi} i t^a \psi + \frac{\partial \mathcal{L}}{\partial (D_\mu \psi)} i t^a (D_\mu \psi) + \cdots + \frac{\partial \mathcal{L}}{\partial F^a_{\mu\nu}} \delta F^a_{\mu\nu} + \frac{\partial \mathcal{L}}{\partial D_\sigma F^a_{\mu\nu}} \delta D_\sigma F^a_{\mu\nu} + \cdots = 0. \tag{2.47}$$

Because of this condition and the transformation law of the gauge field $A^a_\mu(x)$ (2.35), which involves a term linear in $\partial_\mu \alpha^a(x)$, which is clearly not gauge invariant the Lagrangian $\mathcal{L}$ could not depend explicitly on A^a_μ. This is why a mass term $-\frac{1}{2}m^2 A^a_\mu A^{b\mu}$ is ruled out and the "'particles"' which represent the gauge field, so-called gauge-bosons[19], have to be massless in all locally invariant gauge theories.[20]
Parity conservation, Lorentz invariance and the fact that for any massless particle of unit spin the Lagrangian should contain a dynamic factor for the gauge field quadratic in $\partial_\mu A^a_\nu - \partial_\nu A^a_\mu$ combined with the knowledge that because of our non-abelity not A^a_μ or $\partial_\nu A^a_\mu$ but $F^a_{\mu\nu}$ transforms properly under local gauge transformations (2.31) dictates the form of the part of the Lagrangian proportional to $F^a_{\mu\nu}$ as

$$\mathcal{L}_F = -\frac{1}{4} g_{ab} F^a_{\mu\nu} F^{b\mu\nu} \tag{2.48}$$

where g_{ab} is a constant matrix.
One can take g_{ab} to be symmetric and must be taken to be real in order to give a real Lagrangian. This part of the Lagrangian must be invariant for itself and hence satisfy (2.47), because it only depends on $F^a_{\mu\nu}$:

$$0 = \frac{\partial \mathcal{L}_F}{\partial F^a_{\mu\nu}} \delta F^a_{\mu\nu} = -\frac{1}{2} g_{ab} F^b_{\mu\nu} \delta F^a_{\mu\nu} \tag{2.49}$$

$$= \left(-\frac{1}{2} g_{ab} F^b_{\mu\nu}\right) \left(-f^{acd} F^{c\,\mu\nu}\right) \tag{2.50}$$

$$= g_{ab} F^a_{\mu\nu} f^{bcd} F^{b\,\mu\nu} \tag{2.51}$$

where $\delta F^a_{\mu\nu}$ is taken from (2.37) and in the last line some relabeling of indices was done (justified by the symmetry condition on g_{ab}).
This condition can be written independent of any restrictions on the field strength tensors as

$$g_{ab} f^{bcd} = -g_{cb} f^{bad}. \tag{2.52}$$

One can show that with the right rescalings of the gauge fields and the generators we can always take $g_{ab} = \delta_{ab}$ and hence write

$$\mathcal{L}_F = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu}. \tag{2.53}$$

[19] e.g. the photon for the QED gauge field or as will be discussed later the gluon for the QCD gauge field

[20] There are exceptions in the weak interaction theory where the $W^\pm$ and the Z bosons were shown in experiment to have mass. This can be explained by the Higgs mechanism which is not subject of this seminar and will henceforth be ignored.

2.2.4 Conservations laws of the Yang-Mills Lagrangian

The equations of motion of the gauge field A^a_μ of the Yang-Mills Lagrangian (2.45) have the form

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^a_\nu)} = \frac{\partial \mathcal{L}}{\partial A^a_\nu} \tag{2.54}$$

$$\implies -\partial_\mu F^{a\,\mu\nu} = \underbrace{-F^{c\,\nu\mu} f_b^{ca} A^b_\mu - i\frac{\partial(\bar\psi(i\not{D}-m)\psi)}{\partial D_\nu \psi} t^a \psi}_{:=\mathcal{J}^{a\,\nu}} \tag{2.55}$$

$$\implies \partial_\mu F^{a\,\mu\nu} = -\mathcal{J}^{a\,\nu} \tag{2.56}$$

and one can easily verify by plugging the definition of $F^{a\,\mu\nu}$ in (2.37) into (2.56) that $\mathcal{J}^{a\,\nu}$ is conserved:

$$\partial_\nu \mathcal{J}^{a\,\nu} = 0. \tag{2.57}$$

One immediately realizes that gauge invariance is not guaranteed, since all appearing derivatives are the standard partial derivatives. If we rewrite (2.56) in terms of the covariant derivative to make it gauge invariant and write the last term as a commutator, i.e. as an object of the adjoint representation and hence in terms of structure constants of our algebra, we obtain:

$$D_\lambda F^{a\,\mu\nu} = \partial_\lambda F^{a\,\mu\nu} - g f_b^{ac} A^b_\lambda F^{a\,\mu\nu}. \tag{2.58}$$

We conclude from (2.55) and (2.56) that with the covariant derivative the current $\mathcal{J}^{a\,\nu}$ is just the current of the matter fields:

$$D_\mu F^{a\,\mu\nu} = -\mathcal{J}^{a\,\nu} \tag{2.59}$$

$$\implies \mathcal{J}^{a\,\nu} = -i\frac{\partial(\bar\psi(i\not{D}-m)\psi)}{\partial D_\nu \psi} t^a \psi. \tag{2.60}$$

This is gauge invariant since $\mathcal{L}_M = (\bar\psi(i\not{D}-m)\psi)$ is.
Differentiating (2.59) and using

$$[D_\nu, D_\mu]\, F^{a\,\rho\sigma} = -f^a_{cb} F^c_{\nu\mu} F^{b\,\sigma\rho} \tag{2.61}$$

we find that $\mathcal{J}^{a\,\nu}$ satisfies the conservation law

$$D_\nu \mathcal{J}^{a\,\nu} = 0 \tag{2.62}$$

which in contrast to (2.57) is now gauge invariant.
There appears another striking analogy to general relativity which should be mentioned to underline the fact that gauge theories as well as GR are theories based on differential geometry: In GR the curvature tensor $R(X,Y)$ satisfies the so called 2$^{\text{nd}}$ Bianchi identity[21]

[21] in component notation with two contracted indices

$$D_\mu R_{\nu\gamma} + cycl. = 0 \tag{2.63}$$

and our field tensor $F^a_{\mu\nu}$ satisfies

$$D_\mu F^a_{\nu\gamma} + cycl. = 0 \tag{2.64}$$

as can easily be shown using the Jacobi identity for the structure constants.

2.3 The Wilson loop

In the sections 2.1 and 2.2 we defined a comparator $U(y,x)$ to be able to compare the fields $\psi(x)$ and $\psi(y)$. The assumption we made was $y = x + \epsilon n$ and we let $\epsilon \to 0$. If we try to drop this restriction and consider finite separated points we run into problems with the expansion (2.6) and as a consequence we are neither able to define a covariant derivative nor to calculate anything else.
There is a astonishing analogy between gauge theories and general relativity which can be seen if one introduces the concept of fibre-/vector bundles[22] and there one can show that the comparator depends on $F_{\mu\nu}$[23] which determines the kinematic of A_μ. So in general the comparator depends on the path taken.

Our aim is to construct a comparator that makes sense not only for infinitesimal separations but also for finite ones. In our infinitesimal approach we saw that this comparator is by definition a function of the connection $A_\mu(x)$ and it needs to transform according to (2.4).
In the Abelian case we define the so-called *Wilson line* to be

$$U_P(z,y) = \exp\left[-ie \int_P dx^\mu A_\mu(x)\right] \tag{2.65}$$

where the path P is taken from y to z. Again in analogy to section 2.1 define from the definition of the Wilson line a closed loop called *Wilson loop*

$$U_P(y,y) = \exp\left[-ie \oint_P dx^\mu A_\mu(x)\right] \tag{2.66}$$

$$= \exp\left[-i\frac{e}{2} \int_\Sigma d\sigma^{\mu\nu} F_{\mu\nu}\right] \tag{2.67}$$

[22] In differential geometry one can show that a parallel transport depends on the path taken since a parallel transport is nothing else than a continuous map between tangent spaces at different points on the manifold, e.g. in general relativity. In gauge theories there is a striking analogy: the gauge fields "'live"' in vector bundles,i.e. families of vector spaces parametrized by points of a topological space and the fields ψ "'live"' in Minkowski space time. The gauge invariant fields we are examining therefore "'live"' in the direct sum of these spaces and since the vector spaces of the vector bundle are of the same dimensionality there exist isomorphisms between them which contain the important informations. These isomorphisms are represented by the comparators which are maps that map a field $\psi(x)$, which lives in $M \oplus E$ where E is the vector space of the gauge transformations, to a field $\psi(y)$. So the comparator in general depends on the chosen path in the vector bundle.

[23] which is in the language of differential geometry the curvature of the gauge vector bundle

where we have used Stokes theorem to evaluate the last equality. Since $F_{\mu\nu}$ is gauge invariant in the abelian case the comparator is also invariant.
If we now try to generalize this result to the non-Abelian case we will surely run into problems due to the non-commutativity of the generators that generate the transformations on the manifold, which will surely enter our expression via the requirement for the transformation law for the Wilson line.
The method to avoid these complications is to consider a path ordering of the whole expression. We choose a parameter s on our path P running from 0 at $x = y$ to 1 at $x = z$. The path ordering orders the paths such that the higher values of s stand to the left and the lowest value to the very right. Our expression for the Wilson line then reads

$$U_P(z,y) = P\left[\exp\left[ig\int d^\mu x A^a_\mu(x)t^a\right]\right] \tag{2.68}$$

$$= P\left[\exp\left[ig\int_0^1 ds\frac{d^\mu x}{ds}A^a_\mu(x(s))t^a\right]\right]. \tag{2.69}$$

In order to prove the correctness of this expression we recall from differential geometry that a vector parallel transported along a path $\gamma(s)$ transforms as

$$\frac{d}{ds}X^i(s) = -\Gamma^i{}_{lk}\gamma^l(s)X^k(s) \tag{2.70}$$

where $\Gamma^i{}_{lk}$ is the Christoffel-connection which defines the parallel transport.
If we transfer this knowledge to our case of gauge transformations we see that our comparator $U(x(s),y)$ has to fulfill a relation similar to (2.70) which is

$$\frac{d}{ds}U_P(x(s),y) = \left(ig\frac{dx^\mu}{ds}A^a_\mu(x(s))t^a\right)U_P(x(s),y) \tag{2.71}$$

in order to transform correctly under gauge transformations (2.22). To see that this is actually true we rewrite (2.71) with the help of (2.34) as

$$\frac{dx^\mu}{ds}D_\mu U_P(x,y) = 0. \tag{2.72}$$

Also we rewrite (2.22) with a gauge transformed field A^V as

$$U_P(z,y,A^V) = V(z)U_P(z,y,A)V^\dagger(z) \tag{2.73}$$

and we know from (2.27) that the gauge field A^a_μ transforms according to

$$D_\mu(A^V)V(x)\psi = V(x)D_\mu(A)\psi. \tag{2.74}$$

What we therefore want to show is that if $U_P(z,y)$ is the solution of (2.72), then it transforms according to (2.73):

$$D_\mu(A^V)V(z)U_P(z,y,A)V^\dagger(y) = V(z)\underbrace{D_\mu(A)U_P(z,y,A)}_{=0\text{ according to (2.72)}}V^\dagger(y) \tag{2.75}$$

$$= 0 \tag{2.76}$$

and since solutions of 1$^{\text{st}}$ order differential equations with fixed boundaries are unique, $U_P(z,y)$ as defined in (2.68) is the unique expression that transforms according to (2.73). If we carelessly step over to a closed Wilson line and mark it as a Wilson loop we run into problems since our line is not gauge invariant as is stated by equation (2.73) for a closed line

$$U_P(y,y) \longrightarrow V(y)U_P(y,y)V^\dagger(y) \tag{2.77}$$

which is due to the fact that the exponent in distinction to the abelian $U(1)$ case contains non-trivial generators of the transformation algebra and hence in general does not commute any longer with the elements of the transformation group which also contain these generators. But for closed lines our comparator should be gauge invariant in order to be physically meaningful. The easiest way to define a gauge invariant quantity is in most cases to take the trace.
The trace is gauge invariant by cyclic invariance of (2.77) and we get

$$\mathrm{Tr}[U_P(y,y)] \longrightarrow \mathrm{Tr}[U_P(y,y)]. \tag{2.78}$$

We define the Wilson loop in the non-abelian case to be the trace of the Wilson line with which we have found the desired gauge invariant quantity that converts matter fields from one point in gauge transformation space to another to make the comparison and hence the derivation of fields at different points on this manifold sensible.

3 The QCD Lagrangian

To describe the strong interactions Gell-Mann and Zweig proposed so-called "'quarks"' as elementary fermionic particles. Their motivation was the fact that with better experimental methods more and more hadrons, i.e. baryons[24] and mesons[25], were found in accelerators and in cosmic radiation. Also experiments, namely Deep Inelastic Scattering experiments[26], let suggest that these hadrons were built up of smaller particles, so-called partons, which were introduced by Richard Feynman. But this model is quite inaccurate and imprecise whereas the quark-gluon picture explaines the spectrum of these particles better. Since one thought of the quarks to be fermionic, mesons, which have integer spin and hence are bosons, were expected to be built up from a quark-antiquark pair and baryons were thought of as being composed of three quarks.
In order to interpret the known quantum numbers, spin, electric charge and angular momentum of hadrons, Gell-Mann and Zweig originally assumed three different types of quarks, called flavors, which they named up-(u), down-(d) and strange-(s) quarks.
Analogously to taking $SU(2)$ group to be the right group to describe the isospin symmetry of protons and neutrons (or of up- and down-quarks) since they have nearly the same mass, it was sensible to assume the quark triplet to transform under $SU(3)$ transformations, since the strange-quark is only little heavier than the other two. The group of these transformations is called $SU(3)_{\text{flavor}}$ group.

[24]gr. $\beta\alpha\rho\upsilon\varsigma$ barys = "heavy"
[25]gr. $\mu\epsilon\sigma o\varsigma$ mesos = "middle"
[26]similar to the Rutherford experiments point particles like electrons, muons and neutrinos were fired on a hadronic target to find the inner structure of it

3.1 The spin-statistic problem

But with one answer come more questions and in this case a problem with spin and statistics. It was discovered when the Δ^{++} particle was investigated. This hadron consists of 3 up-quarks and its wave function in the ground state (i.e. $L = 0$) is

$$\psi_{\Delta^{++}} = \psi_{\text{spin}} \otimes \psi_{\text{flavor}} \otimes \psi_{\text{spatial}} \tag{3.1}$$

but the Δ^{++} looks like

$$\Delta^{++} = u(\uparrow)u(\uparrow)u(\uparrow) \tag{3.2}$$

with all spins up so ψ_{spin} is symmetric and so are ψ_{flavor} (3 up-quarks) and ψ_{spatial}. But the total wavefunction $\psi_{\Delta^{++}}$ needs to be antisymmetric because Δ^{++} is a fermion ($s = \frac{3}{2}$).
The solution was presented by Greenberg in 1964 and extended and clarified by Gell-Mann in 1972. Greenberg introduced a new internal unobserved quantum number. In analogy to the triplet of fundamental colors it was called color since it also appeared in 3 kinds: red, green and blue (and the corresponding anti-colors).

3.2 The QCD-Lagrangian and gluon fields

The transformation law of the $SU(3)$ group is according to (2.31) and (2.27)

$$q \longrightarrow q' = \underbrace{\exp\left[-i\alpha_k \frac{\lambda_k}{2}\right]}_{:=V(x)} q \tag{3.3}$$

where q[27] represents the quark-field.
The group reads

$$SU(3) = \left\{A \in GL(3,\mathbb{C}) | A^\dagger A = 1, \det A = 1\right\} \tag{3.4}$$

and the elements of the tangential space at the identity $\mathbb{1}$, i.e. the generators of the corresponding Lie-algebra are hermitian since

$$\mathbb{1} = A^\dagger A = (\mathbb{1} + i\epsilon B)^\dagger (1 + i\epsilon B) \tag{3.5}$$

$$= \mathbb{1} + i\epsilon \underbrace{\left(B - B^\dagger\right)}_{=0} \tag{3.6}$$

$$\Rightarrow B = B^\dagger \tag{3.7}$$

and because of the unimodularity of $SU(3)$

$$\text{Tr}[\lambda_k] = 0 \tag{3.8}$$

so we are left with $3^2 - 1 = 8$ degrees of freedom. The set of generators λ of the algebra that are usually used are represented by so called Gell-Mann matrices and they read

[27] this is a very compact notation, all summations over the different quantum numbers (spin, flavor,...) carried by the quarks are implicit in this notation

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad .$$

Nambu, Fritsch and Gell-Mann and Leutwyler supposed in analogy to QED that color charges are sources of gauge fields that carry the strong interaction between quarks.
From the general Yang-Mills theory in section 2.2.2 we know that we can write down a local (and hence also global) invariant Lagrangian that reads

$$\mathcal{L} = \bar{q}(x) \left(i\not{D} - m \right) q(x) - \frac{1}{2} \operatorname{Tr} \left[(G_{\mu\nu})^2 \right] \tag{3.9}$$

where the $G^k_{\mu\nu} T^k = G_{\mu\nu}$ are the "'field-strength tensors"' which determine the kinematics of the gluon fields $G^k_\mu(x)$ defined in analogy to the general case by

$$\begin{aligned} G_{\mu\nu} &= D_\nu G_\mu - D_\mu G_\nu & (3.10)\\ &= \partial_\nu G_\mu - \partial_\mu G_\mu - ig_s \left[G_\mu, G_\nu \right] & (3.11) \end{aligned}$$

and the covariant derivative is defined in analogy to (2.34) as

$$D_\mu = \partial_\mu - ig_s G^k_\mu \frac{\lambda_k}{2} \tag{3.12}$$

If we go into a detailed analysis of this Lagrangian and insert all of the definitions we are able to interpret the theoretical results in a physically meaningful way. To do so we write

$$\begin{aligned} \mathcal{L} &= \bar{q}(x) \left(i\not{D} - m \right) q(x) - \frac{1}{2} \operatorname{Tr} \left[(G_{\mu\nu})^2 \right] & (3.13)\\ &= \bar{q}(x) \left(i\not{\partial} + g_s \not{G}^k(x) \frac{\lambda_k}{2} - m \right) q(x) - \frac{1}{2} \operatorname{Tr} \left[\left(\partial_\nu G_\mu - \partial_\mu G_\mu - ig_s \left[G_\mu, G_\nu \right] \right)^2 \right] & (3.14) \end{aligned}$$

$$\left.\begin{aligned} =&\bar{q}(x) \left(i\not{\partial} - m \right) q(x) - \frac{1}{2} \operatorname{Tr} \left[(\partial_\nu G_\mu - \partial_\mu G_\mu)^2 \right] && \longleftarrow \text{ kinetic terms}\\ &+ g_s \bar{q}(x) \not{G}^k(x) \frac{\lambda_k}{2} q(x) && \longleftarrow \text{ quark-gluon coupling}\\ &+ ig_s \operatorname{Tr} \left(\partial_\nu G_\mu - \partial_\mu G_\nu \right) \left[G_\mu, G_\nu \right] && \longleftarrow \text{ 3 gluon coupling}\\ &+ \frac{1}{2} g_s^2 \operatorname{Tr} \left[G_\mu, G_\nu \right]^2 && \longleftarrow \text{ 4 gluon coupling} \end{aligned}\right. . \tag{3.15}$$

A The Wilson loop in consistent derivation

One of the most interesting facts about QCD is that quarks and gluons never have been observed as single particles. They seem to appear only in bound states and this phenomenon is called *confinement*. One deduces from this observation that the interparticle potential of quarks and gluons has to increase with increasing distance, in contrast to QED where the potential between charged particles goes as $\frac{1}{r}$. One expects a potential that is proportional to r or even higher orders. To show this confinement we consider particles in the high-mass limit[28] in which they become static and compute their evolution with time. We do this first in the abelian case where it is easier to find an appropriate expression for the Wilson loop and secondly generalize this to the non-abelian case.
This section will require some familiarity with Feynman path integrals which were covered in the talk of last week. Also some knowledge of quantum mechanics and the action principle will be of good use.
We consider heavy particles whose action is given by the integral over Minkowski spacetime of (2.45). We couple these particles located at $\vec{x}$ and $\vec{y}$ at $t = 0$ to a gauge potential in a gauge invariant way, i.e. we consider the following state:

$$|\Psi_{\alpha\beta}(\vec{x},\vec{y})\rangle = \bar{\psi}^{(Q)}_{\alpha}(\vec{x},0)U(\vec{x},0;\vec{y},0)\psi^{(Q)}_{\beta}(\vec{y},0)\,|0\rangle \tag{A.1}$$

where $|0\rangle$ denotes a general ground state[29] and $U(\vec{x},0;\vec{y},0)$ is the comparator which compares the field ψ in x and y defined as a pure phase by

$$U(\vec{x},t;\vec{y},t) = \exp\left[ie\int_{\vec{x}}^{\vec{y}} dz^i A_i(\vec{z},t)\right]. \tag{A.2}$$

Our state (A.1) is no eigenstate of the Hamiltonian of our system since our Hamiltonian includes terms proportional to the dynamics of our particles and our state does not include any information about the dynamics. However, we can project it onto the space of eigenstates using the propagation of the state. A general propagator is defined by the amplitude

$$K(\vec{x}',t;\vec{x},0) = \langle x'|\exp\left[-iHt\right]|x\rangle\,. \tag{A.3}$$

Since $|x\rangle$ is not an eigenstate of H consider the spectral decomposition

$$K(\vec{x}',t;\vec{x},0) = \sum_n \langle x'|\,n\rangle\,\langle n|\,x\rangle\exp\left[-iE_n t\right] \tag{A.4}$$

where $|n\rangle$ are the eigenstates of H with the corresponding eigenvalues E_n. We can extract the lowest energy state E_0 by studying euclidean times[30] in the infinite limit:

$$K(\vec{x}',-iT;\vec{x},0) \xrightarrow[T\to\infty]{} \langle x'|\,0\rangle\,\langle 0|\,x\rangle\exp\left[-E_0 T\right]. \tag{A.5}$$

[28]i.e. fix its spatial components
[29]not necessarily the vacuum
[30]$t \to -iT$

Following this strategy we must first compute the propagation amplitude of our state (A.1) in the heavy mass limit and then study its behavior for large euclidean times. Propagations of states are in general described by Green functions in the Feynman path integral method, i.e. in our case by

$$\begin{aligned} G_{\alpha'\beta',\alpha\beta}(\vec{x}',\vec{y}';\vec{x},\vec{y};t) &= \langle \Psi_{\alpha'\beta'}(\vec{x}',\vec{y}',t)|\ \Psi_{\alpha\beta}(\vec{x},\vec{y},0)\rangle \\ &= \langle 0|\, T\left(\bar{\psi}^{(Q)}_{\beta'}(\vec{y}',t)U(\vec{y}',t;\vec{x}',t)\psi^{(Q)}_{\alpha'}(\vec{x}',t)\bar{\psi}^{(Q)}_{\alpha}(\vec{x},0)U(\vec{x},0;\vec{y},0)\psi^{(Q)}_{\beta}(\vec{y},0)\right)|0\rangle \end{aligned} \tag{A.6}$$

where T is the time ordering operator. If we consider this in the heavy mass limit our particles are static and the propagation amplitude becomes

$$G_{\alpha'\beta',\alpha\beta}(\vec{x}',\vec{y}';\vec{x},\vec{y};-iT) \underset{\substack{T\to\infty\\ m\to\infty}}{\longrightarrow} \delta^{(3)}(\vec{x}-\vec{x}')\delta^{(3)}(\vec{y}-\vec{y}')\quad \underbrace{C_{\alpha'\beta'\alpha\beta}(\vec{x},\vec{y})}_{\substack{\text{overlap of (A.1) and the}\\ \text{ground state of } H}}\quad \exp\left[-E(r)T\right] \tag{A.7}$$

where $E(r)$ is the ground state energy as a function of the particles' separation r.
Equation (A.6) can be represented as a path integral[31]:

$$G_{\alpha'\beta',\alpha\beta}(\vec{x}',\vec{y}';\vec{x},\vec{y};t) = \frac{1}{Z}\int \mathcal{D}A\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\,\mathcal{D}\psi^{(Q)}\,\mathcal{D}\bar{\psi}^{(Q)}\left(\bar{\psi}^{(Q)}_{\beta'}(\vec{y}',t)\ldots\psi^{(Q)}_{\beta}(\vec{y},0)\right)\exp\left[iS\right] \tag{A.8}$$

where Z is a normalization constant and S is the action of the Lagrangian (2.45) for the light particles and the gauge fields plus an equivalent term for heavy mass particles.
We can perform the integration over the Grassmann variables $\mathcal{D}\psi^{(Q)}$ and $\mathcal{D}\bar{\psi}^{(Q)}$[32] and obtain

$$\begin{aligned} G_{\alpha'\beta',\alpha\beta}(\vec{x}',\vec{y}';\vec{x},\vec{y};t) = -\frac{1}{Z}\int \mathcal{D}A\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\,\Big[&S_{\beta\beta'}(y,y';A)S_{\alpha\alpha'}(x,x';A) \\ &-S_{\alpha'\beta'}(x',y';A)S_{\beta\alpha}(y,x;A)\Big] \\ &U(\vec{x},0;\vec{y},0)U(\vec{y}',t;\vec{x}',t)\det K^{(Q)}(A)\exp\left[iS\right] \end{aligned} \tag{A.9}$$

where S is given by (2.45).
$S(z,z';A)$ is the Green function which describes the propagation of a particle in the gauge field A_μ given by

$$\left(i\not{D} - m_Q\right)S(z,z';A) = \delta^{(3)}(\vec{z}-\vec{z}')\delta(t-t') \tag{A.10}$$

and $\det K^{(Q)}(A)$ is given by

$$\det K^{(Q)}(A) = \det\left[\left[i\not{D} - m_Q\right]\delta^{(4)}(x-y)\right] \underset{m_Q\to\infty}{\longrightarrow} const. \tag{A.11}$$

[31] $\mathcal{D}\psi = \prod_\alpha d\psi_\alpha$

[32] see appendix C

Notice that in contrast to the bosonic case mentioned in last weeks seminar the integral is now proportional to this determinant instead of proportional to $1/\sqrt{\det A}$ due to the fact that fermionic fields anti-commute whereas bosonic fields commute. But this factor is canceled by a factor in Z and will henceforth be neglected.
If we restrict our result (A.9) to the heavy mass limit, as mentioned before, the particles become static and as a consequence we could neglect the spatial derivatives in (A.10) and simply write

$$\left(i\gamma^0 D_0 - m_Q\right) S(z,z';A) = \delta^{(4)}(\vec{z}-\vec{z}'). \tag{A.12}$$

This differential equation can be solved by the Ansatz

$$S(z,z';A) = \exp\left[ie\int_{z_0}^{z_0'} dt\, A_0(\vec{z},t)\right]\hat{S}(z-z'). \tag{A.13}$$

$\hat{S}(z-z')$ solves the gauge field independent differential equation

$$\left(i\gamma^0\partial_0 - m_Q\right)\hat{S}(z-z') = \delta^{(4)}(z-z') \tag{A.14}$$

which can be integrated by making a Fourier ansatz for $\hat{S}(z-z')$ and we find

$$\begin{aligned} iS(z,z';A) = \delta^{(3)}(z-z')\exp&\left[ie\int_{z_0}^{z_0'} dt\, A_0(\vec{z},t)\right] \\ &\left[\Theta(z_0-z_0')\left(\frac{1+\gamma_0}{2}\right)\exp\left[-im_Q(z_0-z_0')\right]\right. \\ &\left.+\Theta(z_0'-z_0)\left(\frac{1-\gamma_0}{2}\right)\exp\left[im_Q(z_0-z_0')\right]\right]. \end{aligned} \tag{A.15}$$

Because of the δ-function in (A.15) and the fact that $\vec{x}\neq\vec{y}\;\forall\,\vec{x},\vec{y}$ only the first term in (A.9) can contribute. This δ-function in the propagator of a heavy particle reflects the fact, that this particle cannot propagate in space and is therefore static.
Inserting (A.15) in (A.9) we find that

$$\begin{aligned} G_{\alpha'\beta'\alpha\beta} \xrightarrow[m_Q\to\infty]{} \delta^{(3)}(\vec{x}-\vec{x}')\delta^{(3)}(\vec{y}-\vec{y}')&\left(\tfrac{1+\gamma^0}{2}\right)_{\alpha'\alpha}\left(\tfrac{1-\gamma^0}{2}\right)_{\beta'\beta} \\ \exp\left[-2im_Q t\right]&\left\langle \exp\left[ie\oint dz^\mu\, A_\mu(z)\right]\right\rangle \end{aligned} \tag{A.16}$$

where the line integral extends over a closed rectangle with extension $R=|\vec{x}-\vec{y}|$ and t.
If we go to Euclidean times we have to change the action of (2.45) in the expectation value in (A.16) into its Euclidean counterpart and the replacement $x_0\to -ix_4$ implies $A_0\to iA_4$ since $A_\mu=\partial_\mu\Lambda(x)$ in general and $\frac{\partial x_0}{\partial x_4}=-i \Leftrightarrow \frac{\partial x_4}{\partial x_0}=i$.
In the end of all these consideration we obtain for the action

$$S = \int d^4x \; \left[\bar{\psi}\left(\gamma_\mu D_\mu + m\right)\psi + \tfrac{1}{4}F_{\mu\nu}F_{\mu\nu}\right] \tag{A.17}$$

and hence for the propagator

$$G_{\alpha'\beta',\alpha\beta} \underset{\substack{t\to -iT \\ m_Q\to\infty}}{\longrightarrow} \delta^{(3)}(\vec{x}-\vec{x}')\delta^{(3)}(\vec{y}-\vec{y}')\left(\tfrac{1+\gamma^0}{2}\right)_{\alpha'\alpha}\left(\tfrac{1-\gamma^0}{2}\right)_{\beta'\beta}\exp\left[-2m_Q T\right]\left\langle W_C\left[A\right]\right\rangle_{eucl.} \tag{A.18}$$

where

$$W_C\left[A\right] = \exp\left[ie\oint dz_\mu \; A_\mu(z)\right] \tag{A.19}$$

is the Wilson loop we were looking for and C is the rectangular contour we are integrating over.
To find a potential between our particles we have to study these expressions for large Euclidean times. If we compare (A.18) with the expected expression (A.7) we see that $\exp\left[-2m_Q T\right]$ just reflects the rest mass and we expect that for $F(R)$, representing the overlap of our original state with the ground state, and $V(R)$, representing the interaction energy, we will see the following behavior of our system for large Euclidean times:

$$\left\langle W_C\left[A\right]\right\rangle \equiv W(R,T) \underset{T\to\infty}{\longrightarrow} F(R)\exp\left[-E(R)T\right] \tag{A.20}$$

$$\Longrightarrow \qquad V(R) = -\lim_{T\to\infty}\frac{1}{T}\ln\left[W(R,T)\right]. \tag{A.21}$$

B Dirac formalism

Of all representations of the homogeneous Lorentz group there is one with a special role in physics, introduced by Dirac to describe electrons. It can be used as a general spin-$\frac{1}{2}$ representation for fermions with non-integer spin. A general Lorentz transformation is given by

$$x^\mu \longrightarrow \Lambda^\mu{}_\nu x^\nu + a^\mu \tag{B.1}$$

and homogeneity means we set

$$a^\mu = 0. \tag{B.2}$$

a general representation of these transformations is a set of matrices $D(\Lambda)$ which has to fulfill the multiplication law

$$D(\Lambda)D(\bar{\Lambda}) = D(\Lambda\bar{\Lambda}). \tag{B.3}$$

We want to study the properties of these representations near the identity of the group ($\Lambda^\mu{}_\nu = \delta^\mu{}_\nu$) and can hence write

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \tag{B.4}$$

where $\omega^\mu{}_\nu$ is some antisymmetric matrix:

$$\omega_{\mu\nu} = -\omega_{\nu\mu} \tag{B.5}$$

for which

$$D(\Lambda) = 1 + \frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}. \tag{B.6}$$

$\mathcal{J}^{\mu\nu}$ has to be antisymmetric since $\omega_{\mu\nu}$ is, and examining the Lorentz properties of an infinitesimal transformation $D(1+\omega)$ with $D^{-1}(\Lambda) = D(\Lambda^{-1})$

$$D(\Lambda)D(1+\omega)D^{-1}(\Lambda) \stackrel{!}{=} D(\Lambda(1+\omega)\Lambda^{-1}) \tag{B.7}$$

$$\Leftrightarrow \quad D(\Lambda)\left(1 + \frac{i}{2}\omega_{\sigma\rho}\mathcal{J}^{\sigma\rho}\right)D^{-1}(\Lambda) = 1 + \frac{i}{2}\left(\Lambda\omega\Lambda^{-1}\right)_{\mu\nu}\mathcal{J}^{\mu\nu} \tag{B.8}$$

$$\Rightarrow \quad D(\Lambda)\mathcal{J}^{\sigma\rho}D^{-1}(\Lambda) = \Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma\mathcal{J}^{\mu\nu} \tag{B.9}$$

and taking $D(\Lambda)$ itself as infinitesimal and keeping only terms of $\mathcal{O}(\omega)$ gives

$$D(1+\omega)\mathcal{J}^{\sigma\rho}D(1+\omega)^{-1} = \Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma\mathcal{J}^{\mu\nu} \tag{B.10}$$

$$\Leftrightarrow \quad \left(1 + \frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right)\mathcal{J}^{\sigma\rho}\left(1 - \frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right) = \left(\delta_\mu{}^\rho + \omega_\mu{}^\rho\right)\left(\delta_\nu{}^\sigma + \omega_\nu{}^\sigma\right)\mathcal{J}^{\mu\nu} \tag{B.11}$$

and doing some algebra we obtain

$$i\left[\omega_{\mu\nu}\mathcal{J}^{\mu\nu}, \mathcal{J}^{\sigma\rho}\right] = \omega_\mu{}^\rho\mathcal{J}^{\mu\sigma} + \omega_\nu{}^\sigma\mathcal{J}^{\rho\nu} \tag{B.12}$$

and comparing the coefficients of $\omega_{\mu\nu}$ we find

$$i\left[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\sigma\rho}\right] = \eta^{\nu\rho}\mathcal{J}^{\mu\sigma} - \eta^{\mu\rho}\mathcal{J}^{\nu\sigma} - \eta^{\sigma\mu}\mathcal{J}^{\rho\nu} - \eta^{\sigma\nu}\mathcal{J}^{\rho\mu} \tag{B.13}$$

where $\eta^{\mu\nu}$ is the standard Minkowski metric used to raise or lower indices.
To find a set of matrices $\mathcal{J}^{\mu\nu}$ which fulfills these relations we construct γ^μ-matrices which satisfy *anti*-commutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{B.14}$$

and we carelessly define

$$\mathcal{J}^{\mu\nu} = -\frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right] \tag{B.15}$$

which gives us

$$\left[\mathcal{J}^{\mu\nu}, \gamma^\rho\right] = -i\gamma^\mu\eta^{\nu\rho} + i\gamma^\nu\eta\mu\rho. \tag{B.16}$$

Equation (B.16) implies that the γ^μ transform under Lorentz transformations as a vector, i.e.

$$D(\Lambda)\gamma^\mu D^{-1}(\Lambda) = \Lambda_\nu{}^\mu \gamma^\nu. \tag{B.17}$$

A 4-component field that transforms under Lorentz transformations according to (B.6) with (B.15) is called spinor.
We now choose an explicit 4- dim representation of the γ^μ as

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

where the σ^i are the usual Pauli matrices.
To find a Lorentz invariant action for fermions the simplest way is to construct a Lagrangian which is a Lorentz scalar, i.e. which carries no free indices. The first guess would be

$$\psi^\dagger \psi$$

but this doesn't always work as we shall see.
In general one can divide homogeneous Lorentz transformations in 2 classes: rotations and boosts[33].
Boots can be explicitly represented by

$$\mathcal{J}^{0i} = -\frac{i}{4}\left[\gamma^0, \gamma^i\right] = \frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \tag{B.18}$$

which is obviously not hermitian and so the Lorentz transformations generated by $\mathcal{J}^{0i}$ are not unitary. For rotations the case is simple since rotations are unitary transformations so we get

$$\psi^\dagger \psi \longrightarrow \psi^\dagger \Lambda^\dagger \Lambda \psi = \psi^\dagger \Lambda^{-1} \Lambda \psi = \psi^\dagger \psi \tag{B.19}$$

and our first guess would work.
But with boosts it is a different story, since they are not unitary[34] and the 2^{nd} equality of the last equation does not hold. We define

$$\bar{\psi} \equiv \psi^\dagger \gamma^0 \tag{B.20}$$

and check that for boosts since $(\mathcal{J}^{\mu\nu})^\dagger = -\mathcal{J}^{\mu\nu}$ which anticommutes with γ^0

$$\begin{aligned}
\bar{\psi}\psi \longrightarrow & \psi^\dagger \left(D(\Lambda)^{-1}\right)^\dagger \gamma^0 D(\Lambda)\psi && \text{(B.21)} \\
= & \psi^\dagger \left(1 + \tfrac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right)^\dagger \gamma^0 D(\Lambda)\psi && \text{(B.22)} \\
= & \psi^\dagger \left(1 - \tfrac{i}{2}\omega_{\mu\nu}\left(\mathcal{J}^{\mu\nu}\right)^\dagger\right) \gamma^0 D(\Lambda)\psi && \text{(B.23)} \\
= & \psi^\dagger \left(1 + \tfrac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right) \gamma^0 D(\Lambda)\psi && \text{(B.24)} \\
= & \psi^\dagger \gamma^0 \left(1 - \tfrac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right) D(\Lambda)\psi && \text{(B.25)} \\
= & \psi^\dagger \gamma^0 D(\Lambda^{-1}) D(\Lambda)\psi && \text{(B.26)} \\
= & \bar{\psi}\psi && \text{(B.27)}
\end{aligned}$$

[33]transformations whose generators are not conserved, i.e. do not commute with the energy operator $\mathcal{P}^0 = H$

[34]otherwise they would induce a symmetry transformation and commute with the energy operator

so $\bar{\psi}\psi$ is indeed a Lorentz scalar and since $\gamma^\mu\partial_\mu$ is a Lorentz invariant differential operator we can write down a Lorentz invariant Lagrangian

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi \tag{B.28}$$

which would reproduce us as equations of motion the Dirac equation

$$\left(i\gamma^\mu\partial_\mu - m\right)\psi(x) = 0 \tag{B.29}$$

C Grassmann algebra

Since fermion fields in contrast to boson fields anticommute rather than commute[35] we have to find a set of variables with this property and define an algebra and examine its properties in order to treat them right. This formalism is useful when calculating Green functions in a path integral representation for fermions as done in appendix A.
A set of elements $\eta_1, \ldots, \eta_N$ generate a *Grassmann algebra* if they anticommute, i.e.

$$\{\eta_i, \eta_j\} = 0 \tag{C.1}$$

and it immediately follows that

$$\eta_i^2 = 0 \tag{C.2}$$

A general element of this algebra is hence defined as power series in the generators where we have to exclude terms with 2 equivalent indices due to (C.2):

$$f(\eta) := f_0 + \sum_i f_i\eta_i + \sum_{i\neq j} f_{ij}\eta_i\eta_j + \cdots + f_{12\ldots N}\eta_1\ldots\eta_N \tag{C.3}$$

The integration over Grassmann variables is defined completely by two rules:

$$\int d\eta_i = 0 \tag{C.4}$$

$$\int d\eta_i\,\eta_i = 1 \tag{C.5}$$

The integration measures $d\eta_i$ fulfill the same anticommutation rules stated in (C.1) even with the η_j:

$$\{d\eta_i, d\eta_j\} = \{d\eta_i, \eta_j\} = 0 \tag{C.6}$$

Let us practice these rules with the following integral:

$$I[A] = \int \prod_{l=1}^{N} d\bar{\eta}_l\, d\eta_l\, \exp\left[-\sum_{i,j=1}^{N} \bar{\eta}_i A_{ij}\eta_j\right] \tag{C.7}$$

$$= \int \underbrace{\prod_{l=1}^{N} d\bar{\eta}_l\, d\eta_l}_{:=\mathcal{D}(\bar{\eta}\eta)} \prod_{i=1}^{N} \exp\left[-\bar{\eta}_i \sum_{j=1}^{N} A_{ij}\eta_j\right] \tag{C.8}$$

[35]Pauli's exclusion principle

and in the expansion of the exponent only the terms of order 0 and 1 contribute due to (C.2) and we find

$$I[A] = \int \mathcal{D}(\bar{\eta}\eta) \sum_{i_1 \dots i_N} (1 - \bar{\eta}_1 A_{1i_1} \eta_{i_1})(1 - \bar{\eta}_2 A_{2i_2} \eta_{i_2}) \dots (1 - \bar{\eta}_N A_{Ni_N} \eta_{i_N}) \tag{C.9}$$

Because of the integration rules (C.4) and (C.5) we get the integral

$$I[A] = \int \mathcal{D}(\bar{\eta}\eta) \sum_{i_1 \dots i_N} \eta_{i_1} \bar{\eta}_1 \eta_{i_2} \bar{\eta}_2 \dots \eta_{i_N} \bar{\eta}_N A_{1i_1} A_{2i_2} \dots A_{Ni_N} \tag{C.10}$$

where we have changed the order of the Grassmann variables to get rid of the minus signs. Because the indices i_j have to be different and the product of Grassmann variables is antisymmetric we can write this as

$$I[A] = \int \mathcal{D}(\bar{\eta}\eta) \eta_1 \bar{\eta}_1 \eta_2 \bar{\eta}_2 \dots \eta_N \bar{\eta}_N \underbrace{\sum_{i_1 \dots i_N} \epsilon_{i_1 i_2 \dots i_N} A_{1i_1} A_{2i_2} \dots A_{Ni_N}}_{:= \det A} \tag{C.11}$$

where $\epsilon_{i_1 i_2 \dots i_N}$ is the total antisymmetric tensor.

$$I[A] = \underbrace{\prod_{i=1}^{N} \int d\bar{\eta}_i \, d\eta_i \, \eta_i \, \bar{\eta}_i}_{=1 \text{ due to (C.5)}} \det A = \det A \tag{C.12}$$

More relevant to our case in appendix A is the following integral

$$I_{i_1 \dots i_l i'_1 \dots i'_l}[A] = \int \mathcal{D}(\bar{\eta}\eta) \eta_{i_1} \dots \eta_{i_l} \bar{\eta}_{i'_1} \dots \bar{\eta}_{i'_l} \exp\left[- \sum_{i,j=1}^{N} \bar{\eta}_i A_{ij} \eta_j \right] \tag{C.13}$$

Consider the generating functional (with sources $\{\rho_i\}$ and $\{\bar{\rho}_i\}$ in the Grassmann algebra)

$$Z[\rho, \bar{\rho}] = \int \mathcal{D}(\bar{\eta}\eta) \exp\left[- \sum_{i,j=1}^{N} \bar{\eta}_i A_{ij} \eta_j + \sum_{i=1}^{N} (\bar{\eta}_i \rho_i + \bar{\rho}_i \eta_i) \right] \tag{C.14}$$

$$= \left[\int \mathcal{D}(\bar{\eta}\eta) \exp\left[- \sum_{i,j=1}^{N} \bar{\eta}'_i A_{ij} \eta'_j \right] \right] \exp\left[\sum_{i,j=1}^{N} \bar{\rho}_i A^{-1}_{ij} \rho_i \right] \tag{C.15}$$

where we made the substitutions

$$\eta'_i = \eta_i - \sum_k A^{-1}_{ik} \rho_k \tag{C.16}$$

$$\bar{\eta}'_i = \bar{\eta}_i - \sum_k \bar{\rho}_k A^{-1}_{ki} \tag{C.17}$$

so the integration measures $d\eta$ and $d\bar{\eta}$ are invariant under this substitution and glancing at (C.12) we get

$$Z[\rho,\bar{\rho}] = \det A \exp\left[\sum_{i,j=1}^{N} \bar{\rho}_i A_{ij}^{-1} \rho_i\right] \tag{C.18}$$

which is in analogy to the bosonic case seen in last weeks talk which was

$$Z_{\text{bosonic}}[J] = \frac{(2\pi)^{N/2}}{\sqrt{\det M}} \exp\left[\frac{1}{2}\sum_{i,j} J_i M_{ij}^{-1} J_m\right] \tag{C.19}$$

with source J

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