Exercise sheet VIII

due 29.4.2008.

Problem 1 [*Isotropic harmonic oscillator*]:
Solve Schrödinger's equation for the 3d isotropic harmonic oscillator $V(r) = \frac{1}{2}m\omega^2 r^2$ by making use of the spherical symmetry of the problem.

(i) Assume that the eigenfunctions of the Hamiltonian H are of the form

$$\psi(r,\theta,\phi) = f(r) r^l \exp\left(-\frac{m\omega}{2\hbar} r^2\right) Y_l^m(\theta,\phi) .$$

Rewrite H in spherical coordinates,

$$H = \frac{p_r^2}{2m} + \frac{\mathbf{L}^2}{2mr^2} + V(r)$$

where ($p_r = \frac{\hbar}{i}(\frac{1}{r} + \frac{\partial}{\partial r})$) and deduce a differential equation for $f(r)$.

(ii) Define $u(r) = rf(r)$ and derive the differential equation for u. Note that it is helpful to perform a change of variables, taking $\rho = r/b$, with the oscillator length $b = \sqrt{\hbar/m\omega}$.

(iii) Make a power series ansatz for $f(r) = \sum_{\nu=0} a_\nu \rho^\nu$. Why can the series only contain even powers and only a finite number of them, i.e. $f(r) = \sum_{\nu=0}^{K} a_{2\nu}\rho^{2\nu}$? Derive the recursion relation for the coefficients a_ν and determine the possible energies, $E \equiv E(n,l)$ $(n = \nu + 1)$. Compare to the results obtained earlier in Problem 2, Exercise sheet VI.

Problem 2 [*Lenz-Runge vector*]:
The Hydrogen atom has a dynamical symmetry that is bigger than the obvious SO(3) rotation symmetry of the Hamiltonian

$$H = \frac{\mathbf{p}^2}{2\mu} - \frac{\kappa}{r} .$$

The additional symmetry generators are given by the components of the Lenz-Runge vector

$$\mathbf{J} = \frac{1}{2\mu}\Big(\mathbf{p}\wedge\mathbf{L} - \mathbf{L}\wedge\mathbf{p}\Big) - \frac{\kappa}{r}\mathbf{x} , \tag{1}$$

where

$$\mathbf{L} = \mathbf{x}\wedge\mathbf{p} \tag{2}$$

is the angular momentum operator and we have the usual commutation relations

$$[x_i, p_j] = i\hbar\delta_{ij} .$$

(i) Show that $\mathbf{J}$ commutes with the Hamiltonian H. [*Hint*: Evaluate first the commutation relations between the components of $\mathbf{L}$ and the components of $\mathbf{x}$ and $\mathbf{p}$.] Determine also the commutation relations between the components of $\mathbf{L}$ and the components of $\mathbf{J}$.

(ii) Show that

$$\mathbf{J}\cdot\mathbf{L} = 0\ , \qquad \mathbf{J}^2 = \frac{2H}{\mu}(\mathbf{L}^2 + \hbar^2) + \kappa^2\ . \tag{3}$$

Note that the term proportional to $\hbar^2$ is a 'quantum correction' that is not present in the corresponding classical calculation.

(iii) Determine the commutation relations

$$[J_i, J_j] = -\frac{2H}{\mu}\, i\hbar\varepsilon_{ijk}L_k\ . \tag{4}$$

Thus the generators L_i and J_j form a Lie algebra.

(iv) Restrict the action of $\mathbf{J}$ to the subspace of states where the eigenvalue of H is negative. Then define the generators

$$M_i = \frac{1}{\hbar}L_i\ , \qquad K_j = \sqrt{\frac{\mu}{-2H}}\hbar J_j \tag{5}$$

and show that the generators

$$\mathbf{S} = \frac{1}{2}\Big(\mathbf{M}+\mathbf{K}\Big)\ , \qquad \text{and} \qquad \mathbf{D} = \frac{1}{2}\Big(\mathbf{M}-\mathbf{K}\Big) \tag{6}$$

define two commuting su(2) subalgebras. The total Lie algebra is therefore isomorphic to so(4)=su(2)$\oplus$ su(2).