Group Theory Part 1
basics

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In 2004 I started my graduate studies at ETH Zuerich, Switzerland. Part of the course requirement was a 2 semester course on Quantum Mechanics which happened to be lectured by Prof. Froehlich and was (over large sections) basically a course in applied group theory. Since the latter was new for many of us we struggled and were quite happy when Professor Graf offered a lecture just on that: the basics of group theory and its application for $SU(2)$ and S_n (particularly relevant in Quantum mechanics).

It was an eye-opening course and - thanks to my good friend Thomas Willwacher - lots of fun. Thomas would sit for hours with me, explaining all the little details. It was the best time of my graduate studies in physics and so I hope that my notes will add not only to a better understanding but also contribute to just having fun with the physics in Quantum Mechanics.

1 Context

To my knowledge there are 3 ways to tackle a problem in physics:

1. we can try to solve the differential equations analytically (which often is not possible);
2. we can do numerical calculations using a computer;
3. we can exploit the problems symmetry and use group theory to solve the problem

Even though the 3rd option is ignored in most physics courses we encounter it more often than we think:

- **The Hydrogen Atom:** when we solve the hydrogen atom in quantum mechanics, we find the eigenfunctions for the Hamilton operator H

$$H\psi_{n,l,m,\tilde{m}} = E_n\psi_{n,l,m,\tilde{m}} \tag{1}$$

  3 of those 4 quantisation numbers have a meaning that comes straight from the problem's symmetry with respect to rotation.

- **Fermions & Bosons:**
- **electron wave-function in a crystal:** when we solve the quantum-mechanical problem for a periodic potential $V(\mathbf{x})$ (i.e. a crystal) we have to solve the equation:

$$\underbrace{\left(\frac{p^2}{2m} + V(\mathbf{x})\right)}_{H_{1\mathrm{e}}} \Phi_n(\mathbf{x}) = E_n \Phi_n(\mathbf{x}) \tag{2}$$

  and are thought that the solution to that problem are the so called Bloch functions, which for a one dimensional potential with periodicity R can be written as follows:

$$\Phi_k(x) + \exp(ikx) \cdot u_k(x) \tag{3}$$

  where $k := 2\pi/R$ and $u_k(x)$ is a periodic function. Again this result is directly derived from the problems translational symmetry.
- **band structure**
- **Zeemann effect**

Those are just a few prominent examples. Many more can be made. The point I'm trying to make here is simply that because most problems in physics have a symmetry, their solution reflects that and in order to understand it better it is worth spending some time on trying to understand symmetry and how to express it mathematically - which group theory allows us to do.

So symmetry is relevant in physics (I tried to show that); the language with which we can express symmetry is contained in group theory (I hope that for now you will believe that); my strategy is to introduce basics about group theory and then go on to the basics of representations; after that I'd like to have a look at the groups $SU(2)$ and S_n which are particularly important in QM.

2 Groups and Homomorphisms

Motivation: A representation T on a group G is a homomorphism $T : (G, \circ) \longrightarrow (GL(V), \circ)$, where V is a $\mathbb{C}$-vectorspace. That's the motivation to have a closer look at those structures (groups) and the maps between them, that respect this structure (homomorphisms);

A **group** G is defined as a set of elements $\{a, b, c, ...\}$ for which an operation $\star$ is defined:

$$\begin{aligned} \star : G \times G &\longrightarrow G \\ (g, h) &\longmapsto \star(g, h) \equiv gh \end{aligned}$$

This operation is defined between any 2 elements of the group and it has the following properties:

- **Associativity:** for all $a, b, c \in G$ the following is true: $a(bc) = (ab)c$
- **Unit element:** G contains an element e known as **identity** of unit element such that $ea = a \quad \forall\, a \in G$
- **Inverse element:** for all $a \in G$ there exists a corresponding element $a^{-1} \in G$ (called inverse element such that $a^{-1}a = e$

If the group is such that for all $a, b \in G$ the relation $ab = ba$ holds true than the multiplication is called **commutative** and the group itself is called **Abelian**. A **subgroup** $\tilde{G}$ of G is a non-empty subset $\tilde{G} \subseteq G$ that is itself a group (which means it must be closed!). The unit element $\{e\}$ is a trivial subgroup of any group.

Exercise: Show first, that $aa^{-1} = e$ (rather difficult), with this that $ae = a$ (medium), and finally that the unit element is unique (easy). If you can, try and find an example of a set that fulfills the very similar looking axioms $ae = a$ and $a^{-1}a = e$ that is not a group!

Examples Here are some examples that are relevant for us in the future:

- **Permutation Group S_n:** this is the set of all bijective maps $\pi : \{1, 2, 3, ..., n\} \longrightarrow \{1, 2, 3, ..., n\}$. The product of this group is defined canonically: $(\pi_1 \circ \pi_2)(i) := \pi_1(\pi_2(i))$.
- **General Linear Group $GL(V)$:** for every $\mathbb{R}/\mathbb{C}$ vector-space V the set of all invertible, linear maps $\alpha : V \longrightarrow V$ forms with the canonical multiplication a group called $GL(V)$. If $\dim(V) = n$ is finite, then by choosing a basis in V we can define $GL(n, \mathbb{R}/\mathbb{C}) := \{\text{all invertible real/complex valued } n \times n \text{ matrices}\}$ (this is a consistent definition since $V \simeq \mathbb{R}^n/\mathbb{C}^n$).
- $SU(2) := \{A \in GL(2, \mathbb{C}) : \det A = 1 \text{ and } A^*A = 1\}$

- $SO(3) := \{A \in GL(3,\mathbb{R}) : \det A = 1 \text{ and } A^T A = 1\}$
- $(\mathbb{R}^2, +)$

A counter-example is $(\mathbb{Z}, \cdot)$, wich is **not** a group.

A **homomorphism** ϕ between two groups G and H is a map $\phi : G \longrightarrow H$ that respects the group multiplication. This means that $\phi(ab) = \phi(a)\phi(b)$. If ϕ is bijective, then it is called an **Isomorphism**. The **Kernel** of ϕ is the set of all Elements in G that are mapped onto the unit-element in $e_H \in H$:

$$\mathrm{Ker}\phi := \{g \in G : \phi(g) = e_H\}$$

The **Image** of ϕ, on the other hand, is the set of all elements that is reached by the map:

$$\mathrm{Im}\phi := \{\phi(g) : g \in G\}$$

Exercise: show that (i) $\phi(e_G) = e_H$ and that $\phi(g)^{-1} = \phi(g^{-1})$ (ii) $\mathrm{Ker}\phi$ / $\mathrm{Im}\phi$ are a subgroup of G / H (iii) ϕ injective $\Leftrightarrow \mathrm{Ker}\phi = e_H$

From this it follows easily that G is **isomorphic** to the homomorphism's image $\mathrm{Im}\phi \subseteq H$ if and only if $\mathrm{Ker}\phi = \{e_H\}$ or in short

$$G \simeq \mathrm{Im}\phi \quad \Leftrightarrow \quad \mathrm{Ker}\phi = \{e_H\} \tag{4}$$

Exercise: Let G be a group and $H \subseteq G$ be a subgroup of G. (i) show that

$$\begin{array}{ccccl} g & \overset{\mathcal{A}}{\sim} & \tilde{g} & :\Longleftrightarrow & \exists\, h \in H \text{ such that } g = h\tilde{g} \\ g & \overset{\mathcal{B}}{\sim} & \tilde{g} & & \qquad\qquad\qquad\quad g = \tilde{g}h \end{array}$$

$\overset{\mathcal{A}}{\sim}$ and $\overset{\mathcal{B}}{\sim}$ are equivalence-relations (i.e. that they are reflexiv, symmetric and transitiv) (ii) show that $[g]_{\mathcal{A}} = Hg$ (called **right coset**) and similary that $[g]_{\mathcal{B}} = gH$ (called **left coset**) for all $g \in G$

With this it is easy to see that $G/H := \{[g]_{\mathcal{A}/\mathcal{B}} : g \in G\}$ as the set of all right / left cosets is well defined. At this point it makes sense to try and define canonically a multiplication on G/H by $[g] \cdot [h] := [g \cdot h]$. Unfortunately this multiplication is only well defined if H is a **invariant subgroup** which is a subgroup where right and left cosets coincide.

Exercise: show that if $H \subseteq G$ is an invariant subgroup, the multiplication $[g] \cdot [h] := [g \cdot h]$ on G/H is well defined.

with that it is easy to define the **factor group**: Let G be a group and H be an invariant subgroup; then G/H becomes with the canonical multiplication a group itself called the factor group.

Proposition Let G, H be groups and let $\phi : G \longrightarrow H$ be a homomorphism between these groups; then $\mathrm{Ker}\phi$ is an invariant subgroup;
Proof as we already know, $\mathrm{Ker}\phi$ is a subgroup of G; it remains to proof that $(\mathrm{Ker}\phi) \cdot g = g \cdot (\mathrm{Ker}\phi)$ for all $g \in G$
"$\subseteq$": let $k \in \mathrm{Ker}\phi$; then $k \cdot g = (g \cdot g^{-1}) \cdot k \cdot g = g \cdot \underbrace{(g^{-1} \cdot k \cdot g)}_{\in \mathrm{Ker}\phi}$

Main Theorem Let G, H be groups and let $\phi : G \longrightarrow H$ be a homomorphism between these groups; then $G/\mathrm{Ker}\phi \simeq \mathrm{Im}\phi$
Proof the homomorphism $[g] \overset{\Phi}{\longmapsto} \phi(g)$ is well defined (*exercise*), surjective (trivial) and injective (to proof: $\mathrm{Ker}\Phi = [e_G]$, "$\supseteq$" is trivial, "$\subseteq$" lets consider $[g] \in \mathrm{Ker}\Phi \Longrightarrow \phi(g) = e_H \Longrightarrow g \in \mathrm{Ker}\phi = [e] \Longrightarrow [g] = [e]$ (that $\mathrm{Ker}\phi = [e]$ we know from our previous thoughts about left and right cosets: we know that for an invariant subgroup H we can write for any $g \in G$ that $[g] = gH = Hg$ and since here the invariant subgroup is $\mathrm{Ker}\phi$ we know that $[e] = e\mathrm{Ker}\phi = \mathrm{Ker}\phi$)

summary: We learned that the kernel $\mathrm{Ker}\,\phi$ of a homomorphism ϕ on a group G is always not only a subgroup but an *invariant* subgroup. This allows us not only to define the factor-group $G/\mathrm{Ker}\,\phi$ but show that it is *isomorphic* to the image $\mathrm{Im}\,\phi$.

3 Group-algebra and class-functions

<u>*Motivation*</u> Each representation can be assigned a function $f : G \longrightarrow \mathbb{C}$, which is an element of the group-algebra $\mathbb{C}[G]$. This function f will become central in analyzing the representation itself and it has interesting properties. One of these properties is, that it is constant on the conjugacy classes of G and is hence called a class-function. This is the motivation for us to introduce all this terminology.

Let G be a group; the set $\{f : G \longrightarrow \mathbb{C}\}$ of all complex valued functions on G is called the **group-algebra** [1] $\mathbb{C}[G]$. In $\mathbb{C}[G]$ addition "+" and multiplication "*" are defined point-wise. While the definition of the former is done canonically, the latter is defined by convolution:

$$(f_1 * f_2)(g) = \sum_{h \in G} f_1(g \cdot h^{-1}) \cdot f_2(h) \tag{5}$$

Furthermore the expression:

$$\langle f, h \rangle := \frac{1}{|G|} \sum_{g \in G} \overline{f(g)}\, h(g) \tag{6}$$

defines a **scalar product** on $\mathbb{C}[G]$.

<u>*Exercise:*</u> (i) show that the multiplication on $\mathbb{C}[G]$ is well defined, i.e. that associativity and distributivity apply; (ii) show that the the functions $\delta_g \in \mathbb{C}[G]$ with $\delta_g(h) = 1$ if $h = g$ and $\delta_g(h) = 0$ else, forms a basis in $\mathbb{C}[G]$; (iii) show that $(\delta_h * f)(g) = f(h^{-1}g)$ [2] and $(f * \delta_h)(g) = f(gh^{-1})$; (iv) show that $\delta_g * \delta_h = \delta_{gh}$[3];

The **conjugacy classes** of group G are defined by the equivalence-classes of the equivalence-relation:

$$g \sim \tilde{g} \quad :\Longleftrightarrow \quad \exists h \in G \text{ such that } g = h\tilde{g}h^{-1} \tag{7}$$

Functions $f \in \mathbb{C}[G]$ that are constant on the conjugacy classes are called **class-functions**.

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[1] A $\mathbb{K}$**-algebra** A is a $\mathbb{C}$-vector space with regard to $(A, +)$. In addition there is a multiplication "$*$" defined on $(A, *)$ for which associativity and distributivity apply.

[2] $(\delta_h * f)(g) \overset{\text{Def.}}{=} \sum_{\tilde{h} \in G} \left(\delta_h(g \cdot \tilde{h}^{-1}) \cdot f(\tilde{h}) \right)$; the only non-zero term contributing to the sum is the one for which $g \cdot \tilde{h}^{-1} \overset{!}{=} h$ holds true; the rest is obvious

[3] $\delta_g * \delta_h(\tilde{g}) = \delta_g(\tilde{g}h^{-1})$ is obvious from exercise (iii) by setting $f \equiv \delta_g$; this expression is only non-zero for $\tilde{g}h^{-1} = g$ which completes the proof

Example: In order to get a better feeling for this equivalence relationship, lets have a look at two groups and their conjugacy classes:

- the permutation class S_3
- $\mathbb{R}^2$

Main Theorem The class-functions K of $\mathbb{C}[G]$ form the center[4] $Z(\mathbb{C}[G])$ of G

Proof let $f \in K$ be choosen arbitrarily

$$
\begin{array}{lll}
\text{by definition} & f(hgh^{-1}) = f(g) & \forall\, h, g \in G \\
\overset{\tilde{g} \equiv hg}{\Longleftrightarrow} & f(\tilde{g}h^{-1}) = f(h^{-1}\tilde{g}) & \forall\, h, \tilde{g} \in G \\
\overset{\text{see above exercise}}{\Longleftrightarrow} & f * \delta_h = \delta_h * f & \\
\overset{\text{Span}(\delta_h) = \mathbb{C}[G]}{\Longleftrightarrow} & f * \tilde{f} = \tilde{f} * f & \forall\, \tilde{f} \in \mathbb{C}[G]
\end{array}
$$

[4] The **center** of a group G is the set of all elements that commute with every other element of the group

4 Representations and Schur's Lemmata

Let's start with a number of definitions that will be needed for the rest of this chapter: a **representation** T of a group G on a $\mathbb{C}$-vectorspace V is a homomorphism $T : (G, \circ) \longrightarrow (GL(V), \circ)$. T is called a **unitary representation** on $(V, \langle\cdot,\cdot\rangle)$ if it respects the scalar product:

$$\langle T_g(u), T_g(v)\rangle = \langle u, v\rangle \qquad \forall\ u, v \in V \text{ and } \forall\ g \in G \tag{8}$$

Let now $U \subseteq V$ be a subspace of V; if $T_g(U) \subseteq U\ \forall\ g \in G$, then U is called an **invariant subspace** of V with respect to T. Let now $U \neq \{0\}$; if U contains no invariant subspaces with respect to T except for the trivial ones (i.e. $\{0\}$ and itself), then $T|_U$ is called an **irreducible representation**.

Let T and $\tilde{T}$ be two representation on V and $\tilde{V}$ respectively. They are called **equivalent representations** if there is an Isomorphism $L :\ V \longrightarrow \tilde{V}$ such that $\forall\ g \in G$ the following holds true:

$$L \circ T_g = \tilde{T}_g \circ L \qquad \text{which means that} \qquad \begin{array}{ccc} V & \xrightarrow{L} & \tilde{V} \\ {\scriptstyle T_g}\uparrow & & \uparrow{\scriptstyle \tilde{T}_g} \\ V & \xrightarrow[L]{} & \tilde{V} \end{array} \qquad \text{is commutative} \tag{9}$$

This allows us now to define the **character** χ_T **of a representation** T as:

$$\chi_T(g) := \mathrm{tr}\,(T_g) \tag{10}$$

It is an element of the group-algebra $\mathbb{C}$ and obviously well defined: let $\tilde{T}$ be a representation equivalent to T:

$$\chi_{\tilde{T}}(g) \overset{\text{Def}}{=} \mathrm{tr}\left(\tilde{T}_g\right) \overset{\text{equ. 9}}{=} \mathrm{tr}\left(L \circ T_g \circ L^{-1}\right) = \mathrm{tr}\left(L^{-1} \circ L \circ T_g\right) = \mathrm{tr}\,(T_g) \overset{\text{Def.}}{=} \chi_T(g) \tag{11}$$

The interesting thing is now, that any representation is equivalent to a unitary one if the basis in V is chosen appropriately (without proof). To make things easier we shall from here on assume the following:

- that the representations at hand are unitary (without loss of generality);
- that the groups on which they act are finite (without loss of generality and in order to avoid the rather technical Haar measure);
- that the vector spaces on which they are defined are finite dimensional (unless stated otherwise);

Theorem 1. *Let T be a unitary representation of G on V, where V is a Hilbertspace; Then T can be completely decomposed into irreducible representations T_j*

$$T = T_1 \oplus T_2 \oplus T_3 \oplus ... \tag{12}$$

where the same irreducible representation can appear several times.

Proof. if T is irreducible we are already done; if T is not irreducible then there exists a $W \subseteq V$ which is invariant under T; since T is unitary the following statement holds true for all $g \in G$:

$$\langle w, v\rangle = \langle T_g(w), T_g(v)\rangle \quad \text{for all } w, v \in V \tag{13}$$

Naturally this expression holds true in particular for $w \in W$ and $v \in W^{\perp}$; Since $T_g(w) \in W$ (by virtue of W being an invariant subspace), $T_g(v)$ must be in $W^{\perp}$ (since $\langle w, v\rangle = \langle T_g(w), T_g(v)\rangle = 0$); this means that $W^{\perp}$ is an invariant subspace under T as well; This decomposition can be carried out until $T|_{W \subseteq V}$ is irreducible. □

Even though this decomposition is not unique, the **multiplicity** n_α **of a certain irreducible representation** T_α in this sum is! On the following pages we would like to work our way to a result, that allows us to determine n_α with the help of the charakter of T and T_α according to:

$$n_\alpha = \langle \chi_{T_\alpha}, \chi_T\rangle \tag{14}$$

Schur's Lemmata

Lemma 1. *(a) Let T and $\tilde{T}$ be irreducible representations on V and $\tilde{V}$ respectively; let furthermore $L: V \longrightarrow \tilde{V}$ be linear such that*

$$L \circ T_g = \tilde{T}_g \circ L \quad \forall\, g \in G \tag{15}$$

then $L = 0$ or L is a isomorphism (i.e. T is equivalent to $\tilde{T}$).

(b) Let T be an irreducible representation on V and let $L: V \longrightarrow V$ be linear with

$$L \circ T_g = T_g \circ L \quad \forall\, g \in G \tag{16}$$

then $L = \lambda\, \mathrm{id}_V$.

Proof. (a) **1)** $\mathrm{Ker}(L)$ is invariant with respect to T: have to show that for

$$v \in \mathrm{Ker}(V) \implies T_g(v) \in \mathrm{Ker}(L) \overset{\text{Deffinition}}{\iff} L(T_g(v)) = 0 \tag{17}$$

since $L(T_g(v)) \overset{\text{equ.15}}{=} \tilde{T}_g(L(v)) \overset{v \in \mathrm{Ker}(V)}{=} \tilde{T}_g(0) \overset{\tilde{T}\ \text{linear}}{=} 0$, equ.17 is obvious.

2) $\mathrm{Im}(L)$ is invariant with respect to $\tilde{T}$: have to show that for any $g \in G$

$$v \in V \implies \tilde{T}_g(L(v)) \in \mathrm{Im}(L) \overset{\text{Deffinition}}{\iff} \exists\, w \in V \text{ such that } \tilde{T}_g(L(v)) = L(w) \tag{18}$$

since $\tilde{T}_g(L(v)) \overset{\text{equ.15}}{=} L(T_g(v))$, equ.18 is obvious when choosing $w := T_g(v)$.

3) since for an irreducible representation the only invariant subspaces are $\{0\}$ and the vectorspace itself we can argue as follows

$$\begin{array}{ccc}
\mathrm{Ker}(L) \text{ is invariant w. r. to } T & \wedge & \mathrm{Im}(L) \text{ is invariant w. r. to } \tilde{T} \\
\Downarrow & T/\tilde{T} \text{ irred.} & \Downarrow \\
\{\overbrace{\mathrm{Ker}(L) = \{0\}}^{A} \vee \overbrace{\mathrm{Ker}(L) = V}^{B}\} & \wedge & \{\overbrace{\mathrm{Im}(L) = \{0\}}^{C} \vee \overbrace{\mathrm{Im}(L) = V}^{D}\} \\
 & \Updownarrow &
\end{array}$$

$$\underbrace{(A \wedge C) \;\vee\; (B \wedge C) \;\vee\; (B \wedge D)}_{L=0} \;\vee\; \underbrace{(A \wedge D)}_{L \text{ is Isomorphism}} \tag{19}$$

(b) from equ.(16) it follows, that for any $\lambda \in \mathbb{C}$ the expression $(L - \lambda \cdot \mathrm{id}_V)T_g = T_g(L - \lambda \cdot \mathrm{id}_V)$ holds true as well. With Lemma 1b it follows that $L - \lambda \cdot \mathrm{id}_V = 0$ or that $L - \lambda \cdot \mathrm{id}_V$ is an isomorphism. I now choose λ to be an eigen-value of L. It follows that $L - \lambda \cdot \mathrm{id}_V$ cannot be an isomorphism and that hence $L = \lambda\, \mathrm{id}_V$.

$\square$

Quite naturally the question is now how one can possibly find such an L (as mentioned in the previous Lemma) that commutes with T and $\tilde{T}$. The answer to that question is given in the next theorem

Lemma 2. *(a) Let T and $\tilde{T}$ be irreducible representations of G on V and $\tilde{V}$ respectively; let furthermore $A: V \longrightarrow \tilde{V}$ be linear and*

$$L := \frac{1}{|G|} \sum_{g \in G} \tilde{T}_{g^{-1}} A \, T_g \tag{20}$$

then either $L = 0$ or T is equivalent to $\tilde{T}$.

(b) Let T be an irreducible representation on V and let A and L be as in a); then

$$L = \frac{\operatorname{tr}(A)}{\dim(V)} \cdot \mathrm{id}_V \tag{21}$$

Proof. (a) All we need to show is that $LT_h = \tilde{T}_h L$ for any $h \in G$:

$$\begin{aligned} L\,T_h &\overset{\text{Def}}{=} \left(\frac{1}{|G|} \sum_{g \in G} \tilde{T}_{g^{-1}} A \, T_g \right) T_h = \frac{1}{|G|} \sum_{g \in G} \tilde{T}_{g^{-1}} A \, T_{gh} = \frac{1}{|G|} \sum_{\tilde{g} \in G} \tilde{T}_{h\tilde{g}^{-1}} A \, T_{\tilde{g}} \\ &= \tilde{T}_h \cdot \left(\frac{1}{|G|} \sum_{\tilde{g} \in G} \tilde{T}_{\tilde{g}^{-1}} A \, T_{\tilde{g}} \right) = \tilde{T}_h \, L \end{aligned}$$

With Lemma 1a) the proposition is obvious.

(b) With Lemma 1(b) we know that

$$L = \lambda \, \mathrm{id}_V \quad \Longrightarrow \quad \operatorname{tr}(L) = \lambda \dim(V) \tag{22}$$

From its definition it is obvious that

$$\operatorname{tr}(L) = \frac{1}{|G|} \sum_{g \in G} \operatorname{tr}\left(\tilde{T}_{g^{-1}} A \, T_g \right) = \frac{1}{|G|} \sum_{g \in G} \operatorname{tr}\left(T_g \tilde{T}_{g^{-1}} A \right) = \operatorname{tr}(A) \tag{23}$$

From equ. 22 and equ. 23 that $\lambda = \frac{\operatorname{tr}(A)}{\dim(V)}$ and therefore (Lemma 1(b)) the proposition follows

□

If we now choose A cleverly, we can actually show that the matrix-element-functions $(T)_{ij}$ and $(\tilde{T})_{ij} \in \mathbb{C}[G]$ are orthogonal to each other for non-equivalent representations.

Lemma 3. *(a) Let T and $\tilde{T}$ be irreducible representations of G on V and $\tilde{V}$ respectively; if T is not equivalent to $\tilde{T}$ then*

$$\langle (\tilde{T})_{ij}, (T)_{kl} \rangle = 0 \tag{24}$$

(b) Let T be an irreducible representation on V; then

$$\langle (T)_{ij}, (T)_{kl} \rangle = \frac{1}{\dim(V)} \delta_{ik}\delta_{jl} \tag{25}$$

Proof. (a) let $\mathbf{e}_i$ be basis in V, $\tilde{\mathbf{e}}_j$ basis in $\tilde{V}$, let α and $\beta \in \mathbb{N}$ be arbitrary but fixed; if we know define A as follows

$$\begin{array}{rcl} A:\ V & \longrightarrow & \tilde{V} \\ \mathbf{e}_i & \longmapsto & \delta_{\alpha i}\, \tilde{\mathbf{e}}_\beta \end{array}$$

then it is easy to see that A is linear (projection); with L defined as in Lemma 2(a) we can therefore write:

$$L_{ij} \stackrel{\text{Def.}}{=} \frac{1}{|G|} \sum_{g \in G} \left(\tilde{T}_{g^{-1}} A\, T_g \right)_{ij} = \frac{1}{|G|} \sum_{g \in G} (\tilde{T}_{g^{-1}})_{i\mathbf{k}} (A)_{\mathbf{kl}}\, (T_g)_{\mathbf{l}j} \tag{26}$$

Since $\tilde{T}$ is linear we know that $\tilde{T}_{g^{-1}} = \left(\tilde{T}_g\right)^{-1}$ and since $\tilde{T}$ is also unitary, we know that $\left(\tilde{T}_g\right)^{-1} = \left(\tilde{T}_g\right)^{*} \equiv \overline{\left(\tilde{T}_g\right)}^{\mathrm{T}}$. With this, and knowing that $(A)_{kl} = \delta_{k\beta}\delta_{l\alpha}$, we can continue equ. a:

$$L_{ij} \stackrel{\text{equ. a}}{=} \frac{1}{|G|} \sum_{g \in G} (\overline{\tilde{T}_g})_{\mathbf{k}i}\ \delta_{\mathbf{k}\beta}\ \delta_{\mathbf{l}\alpha}\, (T_g)_{\mathbf{l}j} = \frac{1}{|G|} \sum_{g \in G} (\overline{\tilde{T}_g})_{\beta i}\ (T_g)_{\alpha j} \equiv \langle (\tilde{T})_{\beta i}, (T)_{\alpha j} \rangle \tag{27}$$

from Lemma 2(a) we know that if $\tilde{T}$ and T are not equivalent, then $L = 0$ and therefore especially $(L)_{ij} = 0$ for $i, j \in \mathbb{N}$ which proofs the Lemma;

(b) from Lemma 2(b) we know that

$$L_{ij} = \left(\frac{\mathrm{tr}(A)}{\dim(V)} \mathrm{id}_V \right)_{ij} = \frac{\delta_{\beta\alpha}}{\dim(V)} \delta_{ij} \tag{28}$$

With equ.a the Lemma is proven.

□

This leads us to our key result:

Theorem 2. **(a)** *Let T and $\tilde{T}$ be irreducible representations of G on V and $\tilde{V}$ respectively; if T is not equivalent to $\tilde{T}$ then*

$$\langle \chi_{\tilde{T}}, \chi_T \rangle = 0 \tag{29}$$

(b) *Let T and $\tilde{T}$ be irreducible representations on V which are equivalent; then*

$$\langle \chi_{\tilde{T}}, \chi_T \rangle = 1 \tag{30}$$

Proof. **(a)**

$$\begin{aligned}
\langle \chi_{\tilde{T}}, \chi_T \rangle &\overset{\text{Def. ??}}{=} \frac{1}{|G|} \sum_{g \in G} \overline{\chi_{\tilde{T}}(g)}\ \chi_T(g) && (31)\\
&\overset{\text{Def. ??}}{=} \frac{1}{|G|} \sum_{g \in G} \overline{\mathrm{tr}(\tilde{T}_g)}\ \mathrm{tr}(T_g) && (32)\\
&\overset{\text{Def. ??}}{=} \frac{1}{|G|} \sum_{g \in G} \left\{ \overline{\Sigma_i (\tilde{T}_g)_{ii}}\ \Sigma_j (T_g)_{jj} \right\} && (33)\\
&= \sum_{i,j} \left\{ \frac{1}{|G|} \sum_{g \in G} \overline{(\tilde{T}_g)_{ii}}\ (T_g)_{jj} \right\} && (34)\\
&\overset{\text{Def. ??}}{=} \sum_{i,j} \langle (\tilde{T})_{ii}, (T)_{jj} \rangle && (35)\\
&\overset{\text{equ. ??}}{=} \begin{cases} 0 \text{ for } \tilde{T}, T \text{ not equivalent} \\ \Sigma_{i,j} \frac{1}{\dim(V)} \delta_{ij}\delta_{ij} = \Sigma_i \frac{1}{\dim(V)} = 1 \text{ for } \tilde{T}, T \text{ equivalent} \end{cases} && (36)
\end{aligned}$$

(b) see (a)

□

The cool thing in this last step is that by going from Lemma 3 to the main theorem just proven, we managed to go from $(\dim V)^2$ functions on G to a single class-function on G which still characterizes the irreducible representations well enough to extract how many are contained in a given representation T:

Corollary 1. **(a)** *The irreducible representations of a finite group are finite dimensional.*

(b) *The irreducible representations of an Abelian group are 1-dimensional.*

(c) *Let T be a representation of G on V and T_α irreducible representation of G; the multiplicity n_α of T_α in T is given by:*

$$n_\alpha = \langle \chi_{T_\alpha}, \chi_T \rangle \tag{37}$$

(d) *Let T be a representation of G on V and n_α the multiplicity of the irreducible representations in its decomposition; then the following relation holds true:*

$$\langle \chi_T, \chi_T \rangle = \sum_\alpha n_\alpha^2 \tag{38}$$

Proof. **(a)** Let T be irreducible representation of G on $V \neq \{0\}$ (not necessarily finite-dimensional) with $|G| < \infty$; if we now pick $v \in V$ with $v \neq 0$ arbitrary but fixed and define $W := \operatorname{span}\{T_g(v) : g \in G\}$, then the following statements hold true:

- W is finite dimensional because its generating system is finite.
- $W \neq \{0\}$ and $W \subseteq V$ is invariant under $T \overset{T \text{ irred.}}{\Longrightarrow} V = W$.

This proofs the assertion.

(b) let T be an irreducible representation of G on $V \neq \{0\}$ with G being Abelian; for $h \in G$ arbitrary but fixed it is obvious that $T_g\, T_h = T_{gh} = T_{hg} = T_h\, T_g$ for all $g \in G$; with Lemma 1 we know that $T_h = \lambda\, \mathrm{id}_V$; since T is irreducible (i.e. it has no invariant subspaces except for V and $\{0\}$), it has to be one-dimensional;

(c) as proven at the beginning of this chapter it is possible to write T as a sum of its constituent irred. representations: $T = \bigoplus n_\alpha T_\alpha$. Since $\chi_{T_1 \oplus T_2} = \chi_{T_1} + \chi_{T_2}$ holds true trivially, we can write:

$$\langle \chi_{T_\alpha}, \chi_T \rangle = \langle \chi_{T_\alpha}, \chi_{\bigoplus n_\beta T_\beta} \rangle = \langle \chi_{T_\alpha}, \sum_\beta n_\beta\, \chi_{T_\beta} \rangle = \sum_\beta n_\beta \underbrace{\langle \chi_{T_\alpha}, \chi_{T_\beta} \rangle}_{\delta_{\alpha\beta}} \overset{\text{Theo. 2}}{=} n_\alpha \tag{39}$$

(d) as in **(c)** we can write:

$$\langle \chi_T, \chi_T \rangle = \langle \chi_{\bigoplus n_\alpha T_\alpha}, \chi_{\bigoplus n_\beta T_\beta} \rangle = \sum_{\alpha,\beta} n_\alpha\, n_\beta \underbrace{\langle \chi_{T_\alpha}, \chi_{T_\beta} \rangle}_{\delta_{\alpha\beta}} = \sum_\alpha n_\alpha^2 \tag{40}$$

□

Corollary 2. **(a)** *the characters χ_T of a representation are class functions*

(b) $\chi_T(e) = \dim(V)$

(c) $\chi_{T \oplus \tilde{T}} = \chi_T + \chi_{\tilde{T}}$

(d) $\chi_T(g^{-1}) = \overline{\chi_T(g)}$

Proof. **(a)** exercise - you have to show that for $h \in G$ arbitrary but fixed $\chi_T(g) = \chi_T(h^{-1}gh)$ for all $g \in G$, use definition character and the relation $\mathrm{tr}(ABC) = \mathrm{tr}(CAB)$.

(b) exercise - use $T(e) = \mathrm{id}$ (linearity of T!).

(c) exercise - you have to show that $\mathrm{tr}(T_g \oplus \tilde{T}_g) = \mathrm{tr}(T_g) + \mathrm{tr}(\tilde{T}_g)$ which is easy looking at the definition of $V \oplus \tilde{V}$.

(d) exercise - look at definition of character and use the fact that T_g is unitary (i.e. $T_{g^{-1}} = T_g^{-1} = \overline{T^T}$.

$\square$

Theorem of Peter Weyl

Having derived the orthogonality relations and some simple Corrolaries following from them, we now would like to find out more about irreducible representations, their dimensions, their matrix-elements and their characters. In this quest the **regular representation** T^{reg} will be very helpful

$$T^{\text{reg}} : G \longrightarrow \text{GL}\,(\mathbb{C}(G)) \tag{41}$$

$$g \longmapsto T_g^{\text{reg}} \text{ with } T_g^{\text{reg}}(f) := \delta_g * f \tag{42}$$

This regular representation has a bunch of very interesting properties. Let's start with two of them that are fairly easy to see:

Lemma 4. (a) *For its character $\chi_{T^{\text{reg}}}$ the following relation holds true:*

$$\chi_{T^{\text{reg}}}(g) = \begin{cases} |G| & \text{for } g = e \\ 0 & \text{otherwise} \end{cases} \tag{43}$$

(b) *Another interesting property of T^{reg} is that it contains each and every irreducible representation T^α of G. Furthermore the multiplicity n_α for T^α in T^{reg} is equal to the irreducible representation's dimension d_α.*

(c) *let $C, \tilde{C} \subseteq G$ be conjugation classes in G, arbitrary but fixed; then*

$$\frac{|C|}{|G|} \sum_{\alpha=1}^{k} \overline{\chi_{T^\alpha}(\tilde{C})}\, \chi_{T^\alpha}(C) = \delta_{\tilde{C}C} \tag{44}$$

Proof. **(a)** The first assertion is easy to see: $\chi_{T^{\text{reg}}}(e) \stackrel{\text{Def.}}{=} \text{tr}\,(T^{\text{reg}}(e)) = \text{tr}(\text{id}_{\mathbb{C}(\text{G})}) = \dim(\mathbb{C}[G] = |G|$. To see that for all $g \neq e$ the character is 0, we only have to remember that $\delta_g * \delta_h = \delta_{gh}$. This means that if we choose the set $B = \{\delta_g : g \in G\}$ as a basis in $\mathbb{C}(G))$, any $T^{\text{reg}}(g)$ with $g \in G$ will map each and every basis-element in B onto one and only one element in B again. For $g \neq e$ we know furthermore that no element in B is mapped onto itself. That means that the trace of the matrix describing T_g for $g \neq e$ disappears.

(b) as was shown with the main theorem 1c), the multiplicity n_α for T^α in T^{reg} is:

$$n_\alpha = \langle \chi_{T^\alpha}, \chi_{T^{\text{reg}}} \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_{T^\alpha}(g)}\, \chi_{T^{\text{reg}}}(g) \tag{45}$$

$$= \frac{1}{|G|}\, \overline{\chi_{T^\alpha}(e)}\, \chi_{T^{\text{reg}}}(e) = \frac{1}{|G|}\, d_\alpha\, |G| = d_\alpha \tag{46}$$

(c) without proof

□

With those Lemmata we can now tackle the the main theorem of this section, the **theorem of Peter Weyl:**

Theorem 3. **(a)** *Let T^α be the set of non-equivalent, irreducible representations of G with dimensions d_α ($\alpha = 1, ..., k$); then $B := \left\{ T^\alpha_{ij} : \ \alpha = 1, ..., k \text{ and } i, j = 1, ..., d_\alpha \right\}$ is basis of the group-algebra $\mathbb{C}[G]$.*

(b) *Let $M := \left\{ \chi_{T_\alpha} : \alpha = 1, ..., k \right\}$ be the set of characters belonging to the above set of irreducible representations; then M is basis of the set of class-functions in $\mathbb{C}[G]$, i.e. of $Z(\mathbb{C}[G])$.*

Proof. **(a)** For the proof of this assertion we simply have to calculate $\langle \chi_{T^{\mathrm{reg}}}, \chi_{T^{\mathrm{reg}}} \rangle$ in two different ways:

$$\langle \chi_{T^{\mathrm{reg}}}, \chi_{T^{\mathrm{reg}}} \rangle = \begin{cases} \sum_{\alpha=1}^k n_\alpha^2 \overset{\text{equ. 4}}{=} \sum_{\alpha=1}^k d_\alpha^2 \\ \frac{1}{|G|} \sum_{g \in G} \overline{\chi_{T^{\mathrm{reg}}}(g)} \, \chi_{T^{\mathrm{reg}}}(g) = \frac{1}{|G|} \overline{\chi_{T^{\mathrm{reg}}}(e)} \, \chi_{T^{\mathrm{reg}}}(e) = |G| \end{cases} \tag{47}$$

It is obvious from its definition that $\sum_{\alpha=1}^k d_\alpha^2$ is the number of function in B. Those functions are - according to Schur's Lemmata - all orthogonal to each other and therefore linearly independent. From equ. 47 we also know that $\sum_{\alpha=1}^k d_\alpha^2 = |G|$ which is the dimension of the space those functions live in. By definition they therefore constitute a basis.

(b) The set $\{\delta_{\tilde{C}C} : \ C, \tilde{C}$conjugation classes of $G\}$ is trivially a basis of $Z(\mathbb{C}[G])$. according to Lemma 4 the set $\{\chi_{T_\alpha} : \ l = 1, ..., k\}$ is generating system of the basis which means of $Z(\mathbb{C}[G])$ itself. $\{\chi_{T_\alpha} : \ l = 1, ..., k\}$ is also linearly independent and therefore itself a basis.

□

Corollary 3. **(a)** *A group G has as many non-equivalent, irreducible representations as it has conjugacy classes.*

(b) *An Abelian group G has $|G|$ non-equivalent, irreducible representations .*

Proof. **(a)** In theorem 3b) we were able to show that the characters $\left\{ \chi_{T_\alpha} : \alpha = 1, ..., k \right\}$ belonging to the non-equivalent, irreducible representations of G are a basis for the set of class functions. Therefore there are as many irreducible representations as there are class-functions.

(b) for an Abelian group the there are as many conjugacy classes as the group has elements. With Corollary 3a) the assertion is self-evident.

□

With those important tools in mind we can now come to a quite cool theorem that allows us to explicitly write down the projector of any invariant subspace W_{T^α} of a representation T that contains all of the irreducible representations T^α and is called the **isotypical component**. To be more precise: Let $T : G \longrightarrow \mathrm{GL}(V)$ be a representation of G on V; let T^α be the set of non-equivalent, irreducible representations of G $(\alpha = 1, ..., k)$ with dimensions d_α and multiplicity $n_\alpha = \langle \chi_{T^\alpha}, \chi_T \rangle$; From theorem 1 we know that V can be written as a direct sum of sub-vector spaces $V^i_{T^\alpha}$ on which $T|_{V^i_{T^\alpha}} \simeq T^\alpha$ is irreducible:

$$V = \underbrace{V^1_{T^1} \oplus V^2_{T^1} \oplus ... \oplus V^{n_1}_{T^1}}_{W_{T^1}} \;\oplus\; ... \;\oplus\; \underbrace{V^1_{T^k} \oplus V^2_{T^k} \oplus ... \oplus V^{n_k}_{T^k}}_{W_{T^k}} \tag{48}$$

The sum of all the subvectorspaces $\bigoplus_{i=1,..,n_\alpha} V^i_{T^\alpha}$ that belong to the same irreducible representation T^α is called an isotypical component of T.

Theorem 4. *The projector P_{T^α} with $P_{T^\alpha}(V) = W_{T^\alpha}$ is given by:*

$$P_{T^\alpha} = \frac{d_\alpha}{|G|} \sum_{g \in G} \overline{\chi_{T^\alpha}(g)}\; T(g) \tag{49}$$

Proof. still need to do that □

Knowing the isotypical components of a given representation can be a very useful thing in physics. Let us e.g. imagine that a given quantum mechanical system has a symmetry group G and that therefore the system's Hamilton operator H commutes with the representation $\rho : G \longrightarrow GL(V)$:

$$[\rho_g, H] = 0 \quad \text{for all } g \in G \tag{50}$$

Then it is straight forward to show that the W_{T^i} $(i = 1, ..., k)$ already block-diaganolize H.

To this end let us look at the invariant subspace $U := V^i_{T^\alpha}$ of ρ and the projector $P_U : V \longrightarrow U$. In order to see that P_U commutes with ρ

$$[\rho_g, P_U] = 0 \quad \text{for all } g \in G \tag{51}$$

let us consider a $v \neq 0$ in V and its unique decomposition:

$$v \;=\; \underbrace{u}_{\substack{\Cap \\ U}} + \underbrace{w}_{\substack{\Cap \\ U^\perp}} \tag{52}$$

Since U is an invariant subspace of ρ we know that $u \in U$ implies $\rho(u) \in U$ and since furthermore ρ is unitary[5] we also know that $\langle u, w \rangle = 0$ implies $\langle \rho(u), \rho(w) \rangle = 0$. This can be summarized as follows:

$$\rho(v) \;=\; \underbrace{\rho(u)}_{\substack{\Cap \\ U}} \;+\; \underbrace{\rho(w)}_{\substack{\Cap \\ U^{\perp}}} \tag{53}$$

With this it is trivial to see that $P_U(\rho_g(v)) = P_U(\rho_g(u)) \;+ 0 = \rho_g(u) = \rho_g(P_U(v))$ for any $v \in V$ and any $g \in G$.

If we now consider a second invariant subspace $\tilde{U} = V^j_{T^\beta}$ $(\alpha \neq \beta)$ and the two irreducible representations $T := \rho|_U$ and $\tilde{T} := \rho|_{\tilde{U}}$ along with the linear operator $L := P_U H : \tilde{U} \longrightarrow U$, then Schur's first Lemma is applicable and L must be either an isomorphism (which would contradict our assumption $\alpha \neq \beta$) or equal to 0:

$$P_U H|_{\tilde{U}} = 0 \tag{54}$$

Since this line of reasoning is true for any $U := V^i_{T^\alpha}$ and any $\tilde{U} = V^j_{T^\beta}$ as long as $\alpha \neq \beta$, it is safe to say that H will never map a vector $v \neq 0$ from one isotypical component W_{T^α} onto another one W_{T^β}. But that is just another way of saying that the isotypical components are invariant subspace for H.

[5] any representation can be assumed to be unitary without loss of generality