Statistical Physics
Exercise Sheet 12

Exercise 12.1 Dielectric Susceptibility of Free Electrons

Consider a non-interacting one-dimensional gas of spinless electrons, $\mathcal{H} = \frac{1}{L}\sum_k \frac{k^2}{2m} c_k^\dagger c_k$ (with $\hbar = 1$). We want to evaluate its linear response to an external scalar potential, i.e. a perturbation of the form $\delta\mathcal{H} = -e\int dx\phi(x,t)n^-(x) = -e\int dx\phi(x,t)c^\dagger(x)c(x)$.

a) In order to do so, use the Kubo-formula (section 6.1) for the dielectric susceptibility, and show that it can be written as

$$\begin{aligned}\chi_e(x-x',t-t') &= i\Theta(t-t')\langle\left[\hat{n}^+_H(x,t),\hat{n}^-_H(x',t')\right]\rangle_{\mathcal{H}} \\ &= \sum_q \int \frac{d\omega}{2\pi} \underbrace{\sum_k \frac{f(\epsilon_k)-f(\epsilon_{k+q})}{\omega+\epsilon_k-\epsilon_{k+q}+i\eta}}_{\chi(q,\omega)},\end{aligned} \tag{1}$$

where $f(\epsilon)$ denotes the Fermi function, and the integrand in the second line is the Fourier-transform $\chi(q,\omega)$ of $\chi(x-x',t-t')$.

b) The imaginary part of the so-called Lindhard function $\chi(q,\omega)$ obtained in a) encodes the spectrum of the (charge-)excitations that couple to $\phi(x,t)$. Excitations exist only for regions in the $q-\omega$-plane for which $\Im\chi(q,\omega)\neq 0$. Sketch these regions! (Hint: First show that $\chi(q,\omega)$ can be written as

$$\chi(q,\omega) = \sum_{|k|\leq k_F}\left[\frac{1}{\omega+\epsilon_k-\epsilon_{k+q}+i\eta} - \frac{1}{\omega-\epsilon_k+\epsilon_{k+q}+i0}\right] \tag{2}$$

Take the continuum limit and obtain $\Im\chi(q,\omega)$ using the Dirac identity

$$\frac{1}{x\pm i0} = \mathrm{P}\frac{1}{x}\mp i\pi\delta(x), \tag{3}$$

where P denotes the Cauchy principal value, and integration over x is implied.)

Exercise 12.2 High-Temperature Expansion of the 2D Ising Model

Consider an Ising system on a regular, two-dimensional lattice with energy

$$H = -J\sum_{\langle i,j\rangle}\sigma_i\sigma_j - \sum_i h_i\sigma_i, \tag{4}$$

where $\langle i,j\rangle$ denotes pairs of nearest neighbours.

a) Show that in general $\mathrm{Tr}\sigma_i^{2n+1}X = 0$, where X denotes any combination of observables (except σ_i), and that $\mathrm{Tr}\sigma_i^{2n_i}\sigma_i^{2n_j}\ldots\sigma_k^{2n_k} = 2^N$, where N is the number of sites.

b) Show that

$$\chi_0 := \frac{1}{N\beta^2}\sum_{i,j}\partial_{h_i}\partial_{h_j}\log Z\Big|_{h=0} = 1 + \frac{1}{N}\frac{\mathrm{Tr}\sum_{i\neq j}\sigma_i\sigma_j e^{-\beta H_0}}{\mathrm{Tr}e^{-\beta H_0}}, \tag{5}$$

where $Z = \mathrm{Tr}e^{-\beta H}$ and $H_0 = H\Big|_{h=0}$.

b) Using the relation

$$e^{\beta J\sigma_i\sigma_j} = \cosh(\beta J)(1 + w\sigma_i\sigma_j), \tag{6}$$

$w = \tanh\beta J$, discussed in the lecture, expand the numerator and the denominator of the expression for χ in powers of w. Expand the numerator to second order, and the denominator to zeroth order, assuming that each site on the lattice has q nearest neighbours!