Statistical Physics
Exercise Sheet 11

Exercise 11.1 Gaussian Fluctuations in the Ginzburg-Landau Model

Consider the Ginzburg-Landau model, given by (5.63) and (5.64) in the lecture notes. The goal of this exercise is to study the behaviour of the specific heat as one approaches the critical temperature from above. In order to make the model exactly solvable, we assume that quartic fluctuations are negligible and set $B = 0$. Since we are interested in the specific heat without an external magnetic field, we set $H(\vec{r}) = 0$ (note that the case $H(\vec{r}) \neq 0$ is also exactly solvable, and requires little additional effort). Therefore the free energy functional for a given magnetization m and temperature T is given by

$$F(T, m) \; = \; \frac{1}{2} \int d^d r \, \left\{ A m(\vec{r})^2 + \kappa \big[\vec{\nabla} m(\vec{r}) \big]^2 \right\} ,$$

where $A = a\tau$, with $\tau = \frac{T - T_c}{T_c}$. Here T_c is the critical temperature. By assumption, $\tau > 0$ (otherwise F would be unbounded from below and the quartic term which we ignored above would be necessary). For the calculations we assume our system to be a cube of side length L with periodic boundary conditions on m.

(a) Use the Fourier transform

$$m(\vec{r}) \; = \; \frac{1}{\sqrt{L^d}} \sum_{\vec{q}} m_{\vec{q}} \, e^{i \vec{q} \cdot \vec{r}} ,$$

and compute $F(T, m)$ in the transformed coordinates $\{m_{\vec{q}}\}$. What values of $\vec{q}$ are allowed in the sum? What values of $\vec{q}$ are independent?

(b) Compute the canonical partition function

$$Z(T) \; = \; \int \mathcal{D}m \; e^{-F(T,m)/k_B T}$$

using Gaussian integration. To obtain a well-defined integral introduce an ultraviolet cutoff Λ, and integrate only over $\vec{q}$ satisfying $|\vec{q}| < \Lambda$.
Hint: Rewrite $\mathcal{D}m = \prod_{\vec{q}} dm_{\vec{q}} \, dm_{-\vec{q}}$. What is the meaning of $dm_{\vec{q}} \, dm_{-\vec{q}}$ and where does it come from?

(c) Compute the specific heat capacity c_V in the thermodynamic limit $L \to \infty$. Study its behaviour near the critical temperature $\tau = 0$ as a function of the number of dimensions d.

Exercise 11.2 On the Saddle Point of the Ginzburg-Landau Model

We assume again that we are in the paramagnetic phase $\tau > 0$ and neglect the quartic term of the Ginzburg-Landau functional. We include an external magnetic field H (for simplicity we assume it is parallel to some axis and forget its vector nature), so that the free energy is given by

$$F(T, m) \; = \; \int d^d r \, \left\{ \frac{A}{2} m(\vec{r})^2 + \frac{\kappa}{2} \big[\vec{\nabla} m(\vec{r}) \big]^2 - H(\vec{r}) \, m(\vec{r}) \right\} .$$

The goal of this exercise is to understand the shape of the saddle point m for some external field configurations.

(a) Show that the variational equation for the saddle point m is

$$(A - \kappa\Delta)\, m \;=\; H\,.$$

(b) Take an external magnetic field in the shape of a plane wave

$$H(\vec{r}) \;=\; \mathrm{Re}\, H_0 e^{i\vec{k}\cdot\vec{r}}$$

and, using Fourier transformation, compute the resulting magnetization $m(\vec{r})$. What is the magnetic susceptibility?

(c) Take now

$$H(\vec{r}) \;=\; H_0\, \delta(\vec{r})$$

and compute the magnetization in dimensions $d = 1, 2, 3$. For $d = 2$ the computation requires the use of Bessel functions J_ν and of modified spherical Bessel functions of the second kind K_ν. Look these up in the literature along with any useful integral identities required for the exercise.