Statistical Physics
Exercise Sheet 6

Exercise 6.1 Mixture of Two Ideal Gases in a Harmonic Trap

Consider a mixture of two *different* ideal gases A and B in a harmonic trap. The Hamiltonian is given by

$$\mathcal{H} = \sum_{i=1}^{N_A} \left\{ \frac{p_{A,i}^2}{2m_A} + \frac{D_A}{2} q_{A,i}^2 \right\} + \sum_{i=1}^{N_B} \left\{ \frac{p_{B,i}^2}{2m_B} + \frac{D_B}{2} q_{B,i}^2 \right\}. \tag{1}$$

The system is considered to be isolated, i.e. the microcanonical ensemble is to be used in the following.

a) Calculate the entropy of the system.

b) Find the equilibrium value of E_A, the energy of the gas A in the mixture.

c) Find the spatial density distribution. You will need the following formula:

$$\int_0^{\pi} d\theta \quad \sin^{2m}\theta \cos^{2n}\theta = B(m+1/2, n+1/2), \tag{2}$$

with the Beta-function $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, and the Gamma-function $\Gamma(x)$, which is the analytic continuation of the factorial. It might also be useful to look up some basic properties and identities of the Gamma-function.

Exercise 6.2 Relativistic Ideal Gas

Calculate $\langle E \rangle$ for a relativistic ideal gas, and analyze the low- and high temperature limits. Hint: Use the equipartition law!

Exercise 6.3 Fluctuating Membrane

In this exercise you will investigate the thermal fluctuation of a freely moving membrane. In order to do this, let us first clarify what the relevant degrees of freedom for the fluctuations under consideration are. First of all, we will only consider fluctuations perpendicular to the surface of the membrane. To model the system, we introduce a displacement function, $u(x,y)$, that is defined on the membrane and measures the local displacement of the membrane from its equilibrium position (You can imagine that at zero temperature, the membrane lies entirely in the $x-y$-plane, and that increasing the temperature causes each point on the membrane to fluctuate in the z-direction around $z=0$, and it is precisely this displacement in the z-direction that is measured by $u(x,y)$).
Local displacements in the z-direction change the area of the membrane and therefore also change its energy. We will model the area-dependence of the energy using the Hamiltonian

$$\mathcal{H} = \gamma \int_S d\sigma(x,y), \tag{3}$$

where S is the area in the $x-y$-plane covered by the membrane and $d\sigma(x,y)$ is the local change of area relative to the equilibrium area due to displacements.

a) Find an expression for $d\sigma(x, y)$ in terms of $u(x, y)$ for small displacements and thus show that the Hamiltonian of the membrane is given by

$$\mathcal{H} = \frac{\gamma}{2} \int_S dxdy \quad (\nabla u(x, y))^2 . \tag{4}$$

b) In order to do actual calculations with this Hamiltonian, one has to handle the differential operator first. Find the eigenmodes of ∇ for the case of a quadratic membrane of side length L and periodic boundary conditions, and write the Hamiltonian in terms of these eigenmodes!

b) Now the membrane is put into a heat bath at temperature T. Calculate the mean square displacement $\langle u^2 \rangle$, which is defined by

$$\langle u^2 \rangle = \int dxdy \langle u(x, y)^2 \rangle, \tag{5}$$

where $u(x, y)$ denotes the displacement at the point (x, y). Interpret the result in the limit $L \to \infty$!

Hint: Only take fluctuations on length scales larger than some short distance cutoff a into account to arrive at finite results. Can you interpret the meaning of this cutoff?