Theoretical Physics, Problem Set 12.

FS15 *Hand in: 20.05.15*

1. Entropy of a mixed state

The entropy is a measure of how much a state P is mixed. It is defined as

$$s(P) = -\operatorname{tr}(P \log P).$$

a) Show that the entropy increases when two states are mixed:

$$s(\lambda P_1 + (1-\lambda)P_2) \geq \lambda s(P_1) + (1-\lambda)s(P_2), \quad (0 \leq \lambda \leq 1). \tag{1}$$

Equality holds iff $\lambda = 0, 1$ or $P_1 = P_2$.

Hint: For a convex function f, i.e.

$$\lambda f(x_1) + (1-\lambda)f(x_2) \geq f(\lambda x_1 + (1-\lambda)x_2), \quad (0 \leq \lambda \leq 1),$$

and matrices $A = A^*$, $B = B^*$, it holds

$$\operatorname{tr}(f(B)) \geq \operatorname{tr}(f(A) + (B - A)f'(A))$$

(Klein inequality). Further if f is strictly convex, equality holds iff $A = B$.

b) Show that for density matrices over a unitary space (Hilbert space) of dimension d

$$0 \leq s(P) \leq \log d$$

holds, with minimum exactly for pure states and maximum for $P = \mathbf{1}/d$. Determine the minimum and the maximum.