

In the semiclassical approximation we can describe electron as a particle with the momentum \vec{k} at the position \vec{r} . The time evolution of the position \vec{r} and wave vector \vec{k} are given by the equations of motion

$$\frac{d\vec{r}}{dt} = \vec{v}(\vec{k}) = \frac{\partial \mathcal{E}(\vec{k})}{\partial \vec{k}}$$

$$\frac{d\vec{k}}{dt} = \vec{F} = -e \left(\vec{E}(\vec{r}, t) + \frac{\vec{v}(\vec{k})}{c} \times \vec{B}(\vec{r}, t) \right)$$

Let us introduce the distribution function $f(\vec{k}, \vec{r}, t)$, which determines the number of particles in the infinitesimal phase space volume

$$f(\vec{k}, \vec{r}, t) \frac{d^3 k}{(2\pi)^3} d^3 r$$

We ignore spin here

The total number of particles is

$$N = 2 \int \frac{d^3 k}{(2\pi)^3} d^3 r f(\vec{k}, \vec{r}, t)$$

In equilibrium the distribution function is given by the Fermi distribution

$$f_0(\mathbf{k}, \mathbf{r}, t) = \frac{1}{e^{\frac{\epsilon(\mathbf{k}) - \mu}{T}} + 1}$$

When external fields are present (or temperature gradients) the distribution function is different from f_0 . One can write the general equation for $f(\mathbf{k}, \mathbf{r}, t)$ as

$$\frac{df}{dt} = I(f),$$

where time derivative describes the change of $f(\mathbf{k}(t), \mathbf{r}(t), t)$ due to external fields and collision integral $I(f)$ describes effects of scattering. In general electrons can scatter by impurities, other electrons, phonons etc. Time derivative can be written as

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \vec{r}} \frac{d\vec{r}}{dt} + \frac{\partial f}{\partial \vec{k}} \frac{d\vec{k}}{dt}$$

(3)

Using equations of motion for $\frac{d\vec{r}}{dt}$ and $\frac{d\vec{k}}{dt}$ we obtain the Boltzmann equation

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{r}} + \vec{F} \cdot \frac{\partial f}{\partial \vec{k}} = I(f)$$

The collision integral can be expressed through the scattering probability $W(k, k')$ to scatter a quasiparticle with momentum \vec{k} to \vec{k}' .

In order to scatter particle from the state \vec{k} to \vec{k}' one needs the state \vec{k} to be occupied and the state \vec{k}' to be empty. This gives

the factors $f(k, r, t) (1 - f(k', r, t))$. There

is opposite process of scattering from the state k' to k which is given by

$$W(\vec{k}', \vec{k}) f(\vec{k}', \vec{r}, t) (1 - f(\vec{k}, \vec{r}, t))$$

If time reversal symmetry is present

then $W(\vec{k}', \vec{k}) = W(\vec{k}, \vec{k}')$

Then the collision integral is given by

$$\begin{aligned} I(f) &= - \int \frac{d^3 k'}{(2\pi)^3} W(\vec{k}, \vec{k}') \left[f(\vec{k}, \vec{r}, t) (1 - f(\vec{k}', \vec{r}, t)) - \right. \\ &\quad \left. - f(\vec{k}', \vec{r}, t) (1 - f(\vec{k}, \vec{r}, t)) \right] = \\ &= \int \frac{d^3 k'}{(2\pi)^3} W(\vec{k}, \vec{k}') \left[f(\vec{k}', \vec{r}, t) - f(\vec{k}, \vec{r}, t) \right] \end{aligned}$$

If we substitute here the equilibrium Fermi distribution $f_0(\epsilon(k))$ that depends only on the energy $\epsilon(k)$ and consider elastic scattering where energy is conserved than $I(f_0) = 0$

For small electric field distribution function is not much different from the equilibrium one $f = f_0 + f_1$, $f_1 \ll f_0$

In the lowest order in E the scalar function f_1 can be written for isotropic system as

$$f_1 = (\vec{k} \cdot \vec{E}) \cdot A(\epsilon)$$

Substituting it to collision integral

$$I(f) = \kappa E A(\epsilon) \int W(\theta) (\cos \theta' - \cos \theta) \frac{d\Omega'}{4\pi}$$

where θ' is the angle between \vec{k} and \vec{k}' ,
 θ_1 - between \vec{k}' and \vec{E} and θ between
 \vec{k} and \vec{E} . If $\vec{z} \parallel \vec{k}$ and azimuthal
angle of \vec{k}' is φ' we can rewrite θ_1 as

$$\cos \theta_1 = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos \varphi'$$

Inserting it to the collision integral term
with $\cos \varphi'$ will vanish after $d\varphi'$ integration
and

$$\begin{aligned} I(f_1) &= \kappa \cdot E A(\varepsilon) \int w(\theta') (\cos \theta \cos \theta' - \cos \theta) \frac{d\Omega'}{4\pi} = \\ &= \vec{k} \cdot \vec{E} A(\varepsilon) \int w(\theta') (\cos \theta' - 1) \frac{d\Omega'}{4\pi} = \\ &= -f_1 \int w(\theta) (1 - \cos \theta) \frac{d\Omega}{4\pi} = \\ &= -\frac{f_1}{\tau} \end{aligned}$$

with the relaxation time τ given by

$$\frac{1}{\tau} = \int w(\theta) (1 - \cos \theta) \frac{d\Omega}{4\pi}$$

This is called the relaxation time approximation ⁽⁶⁾

where
$$I(f) = -\frac{(f-f_0)}{\tau}$$

Note that we derived it for the (i) isotropic metal and (ii) elastic scattering. The first assumption doesn't apply for real metals. The second assumption is valid only for some scattering processes (as impurity scattering). Nevertheless we will use the relaxation time approximation for the estimates. In general the Boltzmann equation is the integro-differential equation which is much too difficult to solve for realistic situation

Conductivity

Consider the simplest case of a metal in weak uniform electric field, that is time independent. Then the Boltzmann equation reads

$$e \vec{E} \frac{\partial f}{\partial \vec{k}} = \frac{f - f_0}{\tau}$$

For small \vec{E} we can replace f in the left hand side by f_0 then we obtain

$$e \vec{E} \frac{\partial f_0}{\partial \vec{k}} = \frac{f_1}{\tau}$$

Since f_0 depends only on the electron energy $\mathcal{E}(k)$

$$\frac{\partial f_0}{\partial \vec{k}} = \frac{\partial f_0}{\partial \mathcal{E}} \frac{\partial \mathcal{E}}{\partial \vec{k}} = \vec{v} \frac{\partial f_0}{\partial \mathcal{E}} \quad \text{and}$$

$$f_1 = e \tau (\vec{E} \cdot \vec{v}(k)) \frac{\partial f_0}{\partial \mathcal{E}}$$

The quasiparticle current is

$$\vec{j} = -2e \int \frac{d^3 k}{(2\pi)^3} \vec{v}(k) f(k)$$

It is clear that only $f_1(k)$ contributes

to the current since $f_0(k)$ is even and $v(k)$ - odd function of \vec{k} . Thus we obtain

$$\vec{j} = -e^2 \int \vec{v} (\vec{v} \cdot \vec{E}) \tau \frac{\partial f_0}{\partial \epsilon} N(\epsilon) d\epsilon \frac{d\Omega}{4\pi}$$

where we introduced the density of states $N(\epsilon)$

For low temperatures $\frac{\partial f_0}{\partial \epsilon} = -\delta(\epsilon - \epsilon_F)$ and

$$\vec{j} = e^2 N(\epsilon_F) \tau \int \vec{v}_F (\vec{v}_F \cdot \vec{E}) \frac{d\Omega}{4\pi}$$

performing angular averaging in the isotropic case we obtain

$$\vec{j} = \frac{e^2}{3} N(\epsilon_F) v_F^2 \tau \vec{E}$$

$$\text{or } \underline{\sigma = \frac{e^2}{3} N(\epsilon_F) v_F^2 \tau}$$

Substituting expression for the density of states $N(\epsilon_F) = \frac{K_F m^*}{\pi^2 \hbar^3}$ we obtain the

$$\text{Drude result } \sigma = \frac{e^2 n \tau}{m^*}$$

Impurity scattering

29

For the impurity scattering the relaxation time approximation can be derived in the isotropic case. The scattering probability is given by the Fermi golden rule

$$W(k, k') = \frac{2\pi}{\hbar} n_{\text{imp}} |V_{k'k}|^2 \delta(\epsilon_k - \epsilon_{k'})$$

The relaxation time is

$$\frac{1}{\tau} = \int W(\theta) (1 - \cos\theta) \frac{d\Omega}{4\pi}$$

It is different from the life time

$$\frac{1}{\tau_0} = \int W(\theta) \frac{d\Omega}{4\pi}$$

The factor $(1 - \cos\theta)$ gives more weight to back scattering ($\theta \approx \pi$) compared to forward scattering ($\theta \approx 0$). That's why τ is called transport relaxation time

Assuming defects in the form of point charge Ze we can use the Thomas Fermi model for the screened potential with

$$V_{kk'} = \frac{4\pi Ze^2}{|k-k'|^2 + k_{TF}^2}$$

For very strong screening $k_{TF} \gg k_F$ and

$V_{kk'}$ doesn't $\rightarrow \frac{4\pi Ze^2}{k_{TF}^2}$ and

$$\frac{1}{\tau} \approx \frac{\hbar}{m} n_{imp} N(\epsilon_F) \left(\frac{4\pi Ze^2}{k_{TF}^2} \right)^2$$

and $\rho = \frac{e^2 n \tau}{m}$

Note the quadratic dependence of the resistivity on the charge of the ions

Electron - electron scattering

As we discussed in the lecture on Fermi liquid the scattering rate is given by

$$\frac{1}{\tau} \approx \frac{T^2}{E_F}$$

Then $\rho \approx \frac{n e^2 E_F}{m T^2}$ and

$$\rho \approx \frac{m E_F}{n e^2 \hbar} \left(\frac{T}{E_F} \right)^2$$

Umklapp process

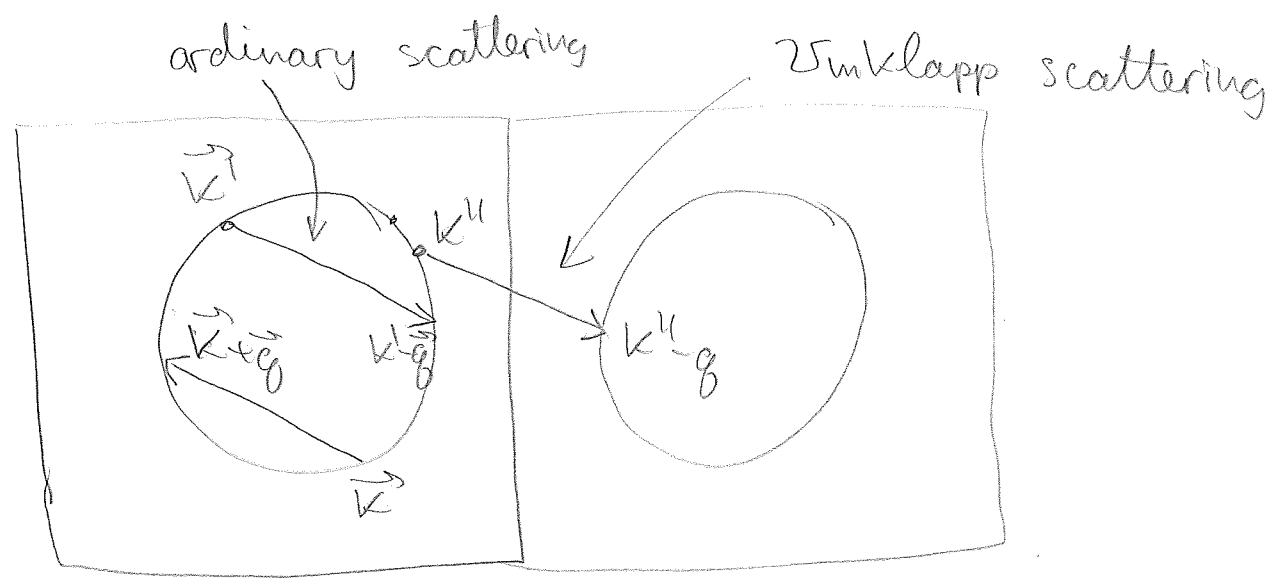
In discussing electron electron interaction one should be more careful. The total momentum is conserved in the electron electron collisions since it is transferred from one electron to another. Then it is not clear what is the origin of a finite resistance. It would be zero if we had full translational invariance. This is not the case since there is underlying crystal lattice

We should take into account that in the crystal \vec{k} is quasi momentum which is defined up to reciprocal lattice vector

$$\vec{k} = \vec{k}' + \vec{G}$$

In this way electron can scatter from one Brillouin zone to another. Then the momentum is transformed to the lattice.

Such processes are called Umklapp processes



Electron phonon interaction

13

The electron phonon interaction is given by $V \sim \sum_F d\sigma U(r)$.

The simplest process of scattering is that when electron emits (or absorbs) phonon.

$$\text{Then } \epsilon_{k'} = \epsilon_k \pm \hbar \omega(k-k')$$

At high temperatures $T \gg \Theta_D$ the number of phonons is given by

$$n(q) = \frac{1}{e^{\frac{\hbar \omega(q)}{T}} - 1} \approx \frac{T}{\hbar \omega(q)}$$

Then $\frac{1}{\tau_{\text{eph}}} \sim \frac{I}{\hbar} \propto n(q)$ and

the resistivity is

$$\rho \sim \frac{m I}{n e^2 \hbar}$$

Low temperatures are more complicated
 Only phonons with $\hbar \omega(q) \lesssim T$ are effective. The electron phonon scattering rate would be given by $d^3q \propto T^3$

But for transport relaxation time we have to take into account $(1 - \cos \theta)$ in the angular integration. But small $\omega(q)$ means

$$q = \frac{\omega}{c} \sim \frac{T}{S} \ll k_F \text{ then}$$

$$\text{typical scattering angle } \theta \approx \frac{q}{k_F} \approx \frac{T}{\Theta_D}$$

$$\text{since } \langle 1 - \cos \theta \rangle \approx \langle \theta^2 \rangle \sim \frac{T^2}{\Theta_D^2}$$

we obtain the relaxation rate at low temperatures

$$\frac{1}{\tau_{\text{eph}}} \sim \frac{T}{\hbar} \left(\frac{T}{\Theta_D} \right)^4 \propto T^5$$

and $\rho \propto T^5$ (Bloch-Grüneisen law)

Umklapp processes may strongly modify it, see Abrikosov textbook

Matthiessen rule

This rule states that the scattering rates of different scattering process can simply be added

$$W(k, k') = W_1(k, k') + W_2(k, k')$$

or
$$\frac{1}{\tau} = \frac{1}{\tau_1} + \frac{1}{\tau_2}$$

and
$$S = \frac{m}{ne^2\tau} = S_1 + S_2$$

It is not a theorem and corresponds to serial coupling of resistors in classical circuit