Lecture 17 Fermi liquid II

Compressibility and sound velocity

The compressibility is defined as

$$\varkappa = -\frac{1}{V}\frac{\partial V}{\partial p}$$, p is the hydrostatic pressure

Longitudinal sound velocity

$$u^2 \equiv C_L^2 = \frac{\partial p}{\partial \rho}$$

Indeed as we discussed for phonons

$$E = \int\left[\frac{\rho}{2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{\lambda(\mathrm{div}\,u)^2}{2}\right] dV$$

and $C_L^2 = \dfrac{\lambda}{\rho}$

But the relative volume change is $\dfrac{\delta V}{V} = \mathrm{div}\,u$,

Thus $\delta E = \dfrac{\lambda}{2}\dfrac{\delta V^2}{V} \Rightarrow$

$\lambda = V\dfrac{\partial^2 E}{\partial V^2}$ and since pressure

$p = -\dfrac{\partial E}{\partial V}$ we obtain $\lambda = -V\dfrac{\partial p}{\partial V} = \rho\dfrac{\partial p}{\partial \rho}$

and $u^2 = \dfrac{\partial p}{\partial \rho}$

Using that the density $\rho = \frac{mN}{V}$ we can rewrite $u^2 = -\frac{V^2}{mN}\frac{\partial P}{\partial V}$

To calculate it let us introduce the chemical potential $\mu = \frac{\partial E}{\partial N}$

Energy is an extensive quantity thus it can be written as $E = V f\left(\frac{N}{V}\right)$

Taking derivatives we obtain

$$\mu = f'\left(\frac{N}{V}\right), \quad P = -f\left(\frac{N}{V}\right) + \frac{N}{V} f'\left(\frac{N}{V}\right)$$

Taking another derivative will produce

$$\frac{\partial \mu}{\partial N} = -\frac{V^2}{N^2}\frac{\partial P}{\partial V}$$

and $\underline{u^2 = \frac{N}{m}\frac{\partial \mu}{\partial N}}$

Since $\mu = \varepsilon_F$ at $T=0$ then

$$\delta\mu = \int f(K_F, K')\,\delta n' \frac{d^3K'}{(2\pi)^3} + \frac{\partial \varepsilon_F}{\partial K_F}\delta K_F$$

First term is the change in the energy $\varepsilon(p_F)$ due to the change of distribution function.

Since $n = \frac{N}{V} = \frac{K_F^3}{3\pi^2\hbar^3} \Rightarrow \delta N = V\frac{K_F^2 \delta K_F}{\pi^2\hbar^3}$

Important $\delta n'$ is near the Fermi surface and

$$\int f\,\delta n' \frac{d^3k'}{(2\pi)^3} = \int f\,\delta n' \frac{d\Omega' K_F^2 dK_F}{(2\pi)^3} =$$

$$= \frac{\delta N}{8\pi V} \int f(\theta)\,d\Omega = \frac{\delta N}{2V} \langle f(\theta) \rangle_\theta$$

Using $\frac{\partial \varepsilon_F}{\partial K_F} = \frac{K_F}{m^*}$ we obtain

$$\frac{\partial \mu}{\partial N} = \frac{1}{8\pi V} \int f(\theta)\,d\Omega + \frac{\pi^2\hbar^3}{K_F m^* V}$$

Substitution $\frac{1}{m^*} = \frac{1}{m} - \frac{K_F}{(2\pi\hbar)^3} \int f(\theta)\cos\theta\,d\Omega$

gives us

$$\frac{\partial \mu}{\partial N} = \frac{\pi^2\hbar^3}{K_F m V} + \frac{\langle f(\theta)(1-\cos\theta)\rangle_\theta}{2V}$$

Multiplying by $\frac{N}{m} = \frac{K_F^3 V}{3\pi^2\hbar^3 m}$ we arrive

$$u^2 = \frac{K_F^2}{3m^2} + \frac{1}{3m}\frac{K_F^3}{(2\pi\hbar)^3} \int f(\theta)(1-\cos\theta)\,d\Omega$$

Note that without the interaction the sound velocity

is $u = \frac{v_F}{\sqrt{3}}$

We can introduce $F(\theta) = \frac{k_F m^*}{\pi^2 \hbar^3} f(\theta) = N(\varepsilon_F) f(\theta)$

Then equation for the effective mass can be rewritten as

$$\frac{m^*}{m} = 1 + \langle F(\theta) \cdot \cos\theta \rangle_\theta$$

And the speed of sound

$$u^2 m^* = \frac{k_F^2 m^*}{3m^2} + \frac{k_F^2}{3m} \frac{k_F m^*}{2\pi^2 \hbar^3} \langle f(\theta)(1-\cos\theta) \rangle_\theta$$

$$u^2 m^* = \frac{k_F^2 m^*}{3m^2} + \frac{k_F^2}{3m} \left(\langle F(\theta) \rangle - \langle F(\theta) \cos\theta \rangle \right)$$

Substituting $\langle F(\theta) \cos\theta \rangle = \frac{m^*}{m} - 1$ we obtain

$$u^2 = \frac{k_F^2}{3 m m^*} (1 + \langle F(\theta) \rangle)$$

For the compressibility we can obtain

$$\varkappa = \frac{\varkappa_0}{1 + \langle F(\theta) \rangle} \quad \text{with } \varkappa_0 = \frac{2}{3} n \varepsilon_F$$

Spin susceptibility

Fermi gas.

In a magnetic field due to the electron spin there is the Zeeman term in the energy

$$\varepsilon(k) = \frac{k^2}{2m} - \mu_B \vec{S}\cdot\vec{H}$$

, where μ_B is Bohr magneton, $\mu_B = \frac{e\hbar}{2mc}$

Then the number of electrons with spin along the field and against the field are

$$n_+ = \frac{1}{2}\int_0^{\mu+\mu_B H} N(\varepsilon)\, d\varepsilon$$

$$n_- = \frac{1}{2}\int_0^{\mu-\mu_B H} N(\varepsilon)\, d\varepsilon$$

The total magnetic moment is

$$M = \mu_B(n_+ - n_-) = \frac{1}{2}\int_{\mu-\mu_B H}^{\mu+\mu_B H} N(\varepsilon)\, d\varepsilon \Rightarrow$$

$$M = \mu_B^2 H N(\varepsilon_F)$$

and the Pauli susceptibility $\underline{\chi_p = \mu_B^2 N(\varepsilon_F)}$

To describe the spin dependent interaction in the Fermi liquid we introduce spin indices for the Landau function

$$\delta \mathcal{E}(\kappa, s) = \sum_{\sigma} \int f(\kappa,\sigma,\kappa',\sigma')\, \delta n(\kappa',\sigma')\, \frac{d^3\kappa'}{(2\pi)^3}$$

Then the energy change in the magnetic field

$$\delta \mathcal{E}(\kappa) = -\mu_B \sigma H + \sum_{\sigma'} \int f(\kappa,\sigma,\kappa',\sigma')\, \delta n'(\kappa,\sigma')\, \frac{d^3\kappa'}{(2\pi)^3}$$

$$\delta \mathcal{E}(\kappa) = -\mu_B \vec{\sigma}\vec{H} + \sum_{\sigma'} \int f(\ldots)\, \frac{\partial n_0}{\partial \mathcal{E}'}\, \delta \mathcal{E}(\kappa',\sigma')\, \frac{d^3\kappa'}{(2\pi)^3}$$

We are looking for a solution of the form

$\delta \mathcal{E} = -g\,\mu_B(\sigma \cdot H)$ with some constant g

to be determined.

In the isotropic liquid the exchange interaction has the form

$$f(\kappa,\sigma,\kappa',\sigma') = f(\kappa,\kappa') + \vec{\sigma}\cdot\vec{\sigma}'\, \xi(\kappa,\kappa'),$$

where $\vec{\sigma}$ are Pauli matrices

Since $n_0(p)$ is a step function $\frac{dn_0'}{d\varepsilon'} = -\delta(\varepsilon'-\varepsilon_F)$

Thus $\sum_{\sigma'} \int \left[f(\kappa,\kappa') + \vec{\sigma}\cdot\vec{\sigma}'\,\xi(\kappa,\kappa')\right]\left(-g\mu_B\,\sigma'\cdot H\right) N(\varepsilon_F)\frac{d\Omega}{8\pi}$

Since $Sp\,\vec{\sigma} = 0$, $Sp'(\vec{\sigma}\vec{\sigma}')\vec{\sigma}' = 2\vec{\sigma}$

It simplifies to $-2(\vec{\sigma}\cdot\vec{H})\int \xi(\kappa,\kappa')\,N(\varepsilon_F)\frac{d\Omega}{8\pi}$

and $g = 1 - g\int \xi(\kappa,\kappa')\,N(\varepsilon_F)\frac{d\Omega}{4\pi} =$

$$g = 1 - g\langle Z(\theta)\rangle$$

with $Z(\theta) = \xi(\theta)\,N(\varepsilon_F)$

Then $g = \dfrac{1}{1+\langle Z(\theta)\rangle}$

Susceptibility is $\chi H = \mu_B\,Sp\int \sigma\,\delta n\,\frac{d^3\kappa}{(2\pi)^3}$

$$= \mu_B\,Sp\int \sigma\,\delta\varepsilon\,\frac{\partial n}{\partial\varepsilon}\frac{d^3\kappa}{(2\pi)^3} = 2gH\int\frac{\partial n}{\partial\varepsilon}\frac{d^3\kappa}{(2\pi)^3}$$

$$= \mu_B^2\,\frac{m^*\kappa_F}{\pi^2\hbar^2}\,g = \mu_B^2\,\frac{N(\varepsilon_F)}{1+\langle Z(\theta)\rangle}$$

For He^3 at zero pressure

$\frac{m^*}{m} \simeq 3$, $\langle F(\theta)\rangle \simeq 10$, $\langle Z(\theta)\rangle \simeq -0.5$, $\varkappa \simeq 0.27\,\varkappa_0$

$$\chi \simeq 6.3\,\chi_0$$

Microscopic consideration

For weakly interacting electron gas one can calculate Fermi liquid corrections microscopically

$$H = \sum_{k,s} \varepsilon_k c^+_{ks} c_{ks} + \frac{v}{V} \sum c^+_{k+q,\uparrow} c^+_{k'-q\downarrow} c_{k'\downarrow} c_{k\uparrow}$$

We assumed here contact interaction between particles $v(r,r') = v\,\delta(r-r')$

For small v we can treat interaction perturbatively. $n_{ks} = c^+_{ks} c_{ks} = n^0_{ks} + \delta n_{ks}$

$$E = E_0 + E_1 + E_2 + \ldots$$

$$E_0 = \sum \varepsilon_k n_{ks}$$

$$E_1 = \frac{v}{V} \sum n_{k\uparrow} n_{k'\downarrow}$$

$$E_2 = \frac{v^2}{V^2} \sum \frac{n_{k\uparrow} n_{k'\downarrow} (1 - n_{k+q,\uparrow})(1 - n_{k'-q,\downarrow})}{\varepsilon_k + \varepsilon_{k'} - \varepsilon_{k+q} - \varepsilon_{k'-q}}$$

E_2 describes virtual processes corresponding to a pair of particle-hole excitations

The numerator in E_2 consist of four different terms. Quadratic in n_k term can be combined with E_1 which has the same structure

$$\tilde{E}_1 = E_1 + \frac{\upsilon^2}{V^2} \sum_{k,k',q} \frac{n_{k\uparrow} n_{k\downarrow}}{\varepsilon_k + \varepsilon_{k'} - \varepsilon_{k+q} - \varepsilon_{k'-q}} =$$

$$= \frac{\tilde{\upsilon}}{V} \sum n_{k\uparrow} n_{k\downarrow}$$

with renormalized interaction $\tilde{\upsilon}$

$$\tilde{\upsilon} = \upsilon + \frac{\upsilon^2}{V} \sum \frac{1}{\varepsilon_k + \varepsilon_{k'} - \varepsilon_{k+q} - \varepsilon_{k'-q}}$$

Quartic term $\sum \frac{n_{k\uparrow} n_{k'\downarrow} n_{k+q,\uparrow} n_{k'-q,\downarrow}}{\varepsilon_k + \varepsilon_{k'} - \varepsilon_{k+q} - \varepsilon_{k'-q}}$

vanishes due to symmetry: numerator is symmetric with $k \to k+q$, $k' \to k'-q$ but denominator is antisymmetric

The cubic terms give

$$\tilde{E}^2 = -\frac{\tilde{v}^2}{V^2} \sum \frac{n_{k\uparrow} n_{k'\downarrow} (n_{k+q,\uparrow} + n_{k'-q,\downarrow})}{\varepsilon_k + \varepsilon_{k'} - \varepsilon_{k+q} - \varepsilon_{k'-q}}$$

where we replace v by the renormalized interaction $\tilde{v}$. Excitation spectrum is given by $\varepsilon_\uparrow(k) = \frac{\delta \tilde{E}}{\delta n_i}$

$$\varepsilon_\uparrow(k) = \varepsilon_k + \frac{\tilde{v}}{V} \sum n_{k'\downarrow} -$$

$$- \frac{\tilde{v}^2}{V^2} \sum \frac{n_{k'\downarrow}(n_{k+q,\uparrow} + n_{k'-q,\downarrow}) - n_{k+q,\uparrow} n_{k'-q,\downarrow}}{\varepsilon_k + \varepsilon_{k'} - \varepsilon_{k+q} - \varepsilon_{k'-q}}$$

The Landau function is obtained by differentiating $\varepsilon_\uparrow$ over δn_k. Important terms are

$$\frac{\tilde{v}^2}{V^2} \sum n_{k+q,\uparrow} \frac{n_{k'-q\downarrow} - n_{k'}}{\varepsilon_k + \varepsilon_{k'} - \varepsilon_{k+q} - \varepsilon_{k'-q}}$$

For k on the Fermi surface it transform to

$$-\frac{1}{V} \sum n_{k'_F\uparrow} \frac{\tilde{v}^2}{2} \chi_0 (k'_F - k_F)$$

where X_0 is the Lindhard function

$$X_0(q) = \frac{1}{V}\sum_{k,s}\frac{n_{k+q,s}-n_{k,s}}{(\varepsilon_{k+q}-\varepsilon_k)}$$

and $f_{\uparrow\uparrow}(k_F,k_F') = f_{\downarrow\downarrow}(k_F,k_F') = \frac{\tilde{v}^2}{2}X_0(k_F-k_F')$

$f_{\uparrow\downarrow}$ can be computed in a similar fashion

As a result we obtain

$$f_{\sigma\sigma'}(\theta) = \frac{\tilde{v}}{2}\left[\left(1+\frac{\tilde{v}N(\varepsilon_F)}{4}\right)\left(2+\frac{\cos\theta}{2\sin(\theta/2)}\ln\frac{1+\sin\theta/2}{1-\sin\theta/2}\right)\delta_{\sigma\sigma'}\right.$$
$$\left.-\left(1+\frac{\tilde{v}N(\varepsilon_F)}{4}\right)\left(1-\frac{\sin(\theta/2)}{2}\ln\frac{1+\sin\theta/2}{1-\sin\theta/2}\right)\sigma\sigma'\right]$$

Effective mass is then

$$\frac{m}{m^*} = 1-\frac{1}{30\pi^2}(7\ln 2-1)\left(\frac{m\tilde{v}k_F}{\hbar}\right)^2$$

For more details see Lifshitz & Pitaevsky Volume IX Statistical Physics Part 2.