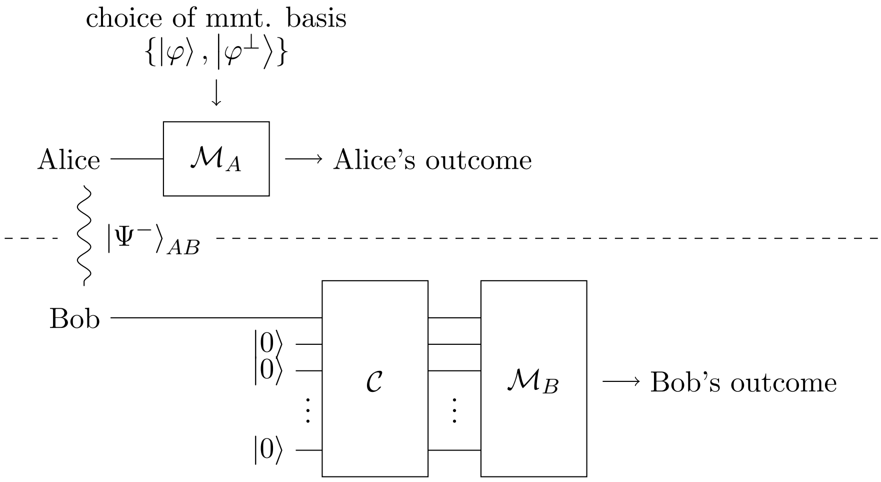
Exercise 1. *Cloning implies signalling*

We have seen in the lecture that no unitary operation (actually: no completely positive trace preserving map) can clone a quantum state. Because in quantum mechanics the evolution of a closed system is unitary, this proves that quantum states cannot be cloned. In this exercise we will see that quantum cloning is incompatible with no-signalling which is believed to be a fundamental principle. This provides us with an argument for the impossibility of cloning that does not rely on any assumption regarding the evolution of quantum systems.

A n-fold quantum cloning machine implements the operation

$$\mathcal{C} : |\psi\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle \;\longmapsto\; |\psi\rangle \otimes |\psi\rangle \otimes \cdots \otimes |\psi\rangle$$

for all possible inputs $|\psi\rangle$, i.e. it produces n copies of an arbitrary quantum state. Consider Alice and Bob, two agents who are spatially separated and share a Bell pair $|\Psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(|01\rangle_{AB} - |10\rangle_{AB})$. The no-signalling principle implies that, no matter what Alice does to her part of the Bell state, she cannot influence the measurement statistics on Bob's side. In the following we show that if Bob has a quantum cloning machine then he can detect what measurement Alice carried out on her side. Hence Alice could signal without sending an information carrier to Bob.



(a) Suppose Alice measures in the basis $|\varphi\rangle = \alpha|0\rangle + \beta|1\rangle$, $|\varphi^\perp\rangle = \beta^*|0\rangle - \alpha^*|1\rangle$, where $|\alpha|^2 + |\beta|^2 = 1$. What is the state on Bob's side conditioned on Alice's outcome after her measurement?

(b) Alice can do one out of two measurements, either in the $\{|0\rangle, |1\rangle\}$ basis or in the $\{|+\rangle, |-\rangle\}$ basis. With the help of the cloning machine, explain how Bob can find out about Alice's measurement choice with high probability for large n.

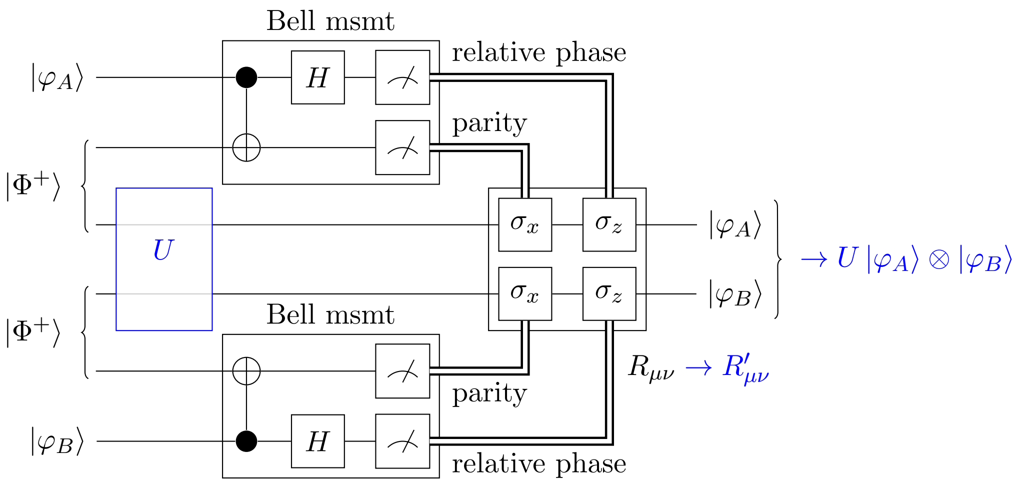
Hint. Bob can perform a measurement on the output of the cloning machine. What measurement would tell him with high probability the measurement choice of Alice?

(c) With the protocol established in (b) Alice can signal one bit, encoded in her choice of measurement, to Bob under the assumption that they shared a maximally entangled state and that Bob has access to a quantum cloning machine. What goes wrong with the procedure if they instead shared n copies of a maximally entangled state, $|\Psi^-\rangle_{AB}^{\otimes n}$, but without the cloning machine?

(d) In fact, the number of bits Alice can signal to Bob making use of a Bell pair and the cloning machine is much larger than one. How can the protocol from (b) be extended to allow Alice to signal an unlimited amount of bits to Bob?

Exercise 2. ***Teleportation through a two-qubit gate***

As shown in the lecture, by a modification of the standard teleportation scheme, one can apply a gate U during the teleporation process. This can be done even for multi-qubit gates. For two qubits, it looks as follows:



Without the U gate, this would simply be ordinary teleportation of two qubits in parallel. In this case, the rotations $R_{\mu\nu} = \sigma_\mu \otimes \sigma_\nu$ that have to be applied to the output qubits depend on the outcome of the Bell measurements as usual:

$$\left|\Phi^+\right\rangle \rightarrow \sigma_\mu/\sigma_\nu = 1 \quad \left|\Phi^-\right\rangle \rightarrow \sigma_\mu/\sigma_\nu = \sigma_x \quad \left|\Psi^+\right\rangle \rightarrow \sigma_\mu/\sigma_\nu = \sigma_y \quad \left|\Psi^-\right\rangle \rightarrow \sigma_\mu/\sigma_\nu = \sigma_z$$

Simply by applying the gate U to the teleportation target beforehand would not give us the desired output $U|\varphi_A\rangle \otimes |\varphi_B\rangle$. Instead the set of output rotations $R_{\mu\nu}$ has to be modified as well.

(a) For a general U, find the set of output rotations $R'_{\mu\nu}$ such that we get the desired output.

(b) What could be the advantages and disadvantages of such a scheme, compared to direct application of U.

(c) Find $R'_{\mu\nu}$ for $U = \text{CNOT}$. *Hint: It might be useful to write* $\text{CNOT} = [|\uparrow\rangle\langle\uparrow| \otimes 1 + |\downarrow\rangle\langle\downarrow| \otimes \sigma_x]$ *and then perform the conjugation on control and target qubits separately.*