Exercise 1. ***Relativistic particle in a constant, uniform magnetic field***

Consider a point particle with mass m, charge q, initial velocity $\mathbf{v}_0$ and initial position $\mathbf{x}_0$, moving in a constant, uniform magnetic field $\mathbf{B} = B\,\hat{\mathbf{e}}_z$, parallel to the z axis. Let the 4-momentum be $p^\mu = (\frac{\mathcal{E}}{c}, \mathbf{p})$.

1. Show that the energy of the particle is constant in time, e.g.
$$\dot{\mathcal{E}} = 0.$$
2. Find the trajectory of the particle.
3. What are the differences between the classical and the relativistic trajectory?

Exercise 2. ***Lorentz transformations for the Electromagnetic field***

a) Prove that under a general Lorentz transformation the $\vec{E}$ and $\vec{B}$ fields transform as follows:
$$\vec{E}' = \gamma\big(\vec{E} + c\,\vec{\beta} \times \vec{B}\big) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{E}), \tag{1}$$
$$\vec{B}' = \gamma\big(\vec{B} - c^{-1}\,\vec{\beta} \times \vec{E}\big) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{B}), \tag{2}$$
where $\vec{\beta} = \vec{v}/c$, $\gamma = (1-\beta^2)^{-1/2}$ and c is the speed of light.

b) Argue what happens to the angle between the $\vec{E}$ and $\vec{B}$ fields under a general boost transformation.

Exercise 3. ***Electrodynamics in a Covariant formalism***

a) Given the electromagnetic field tensor $F^{\mu\nu}$ with components
$$F^{0i} = -E^i, \qquad F^{ij} = -\epsilon^{ijk}B_k, \qquad F^{\mu\nu} = -F^{\nu\mu}, \tag{3}$$
with $\epsilon_{123} = +1$, compute
$$-\frac{1}{4}\,F^{\mu\nu}\,F_{\mu\nu}, \qquad \epsilon_{\mu\nu\rho\sigma}\,F^{\mu\nu}\,F^{\rho\sigma} \tag{4}$$
in terms of the $\vec{E}$ and $\vec{B}$ fields.

b) Show that all the Maxwell equations
$$\partial_t\vec{B} + \vec{\nabla}\times\vec{E} = 0 \tag{5}$$
$$\vec{\nabla}\cdot\vec{B} = 0 \tag{6}$$
are equivalent to
$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0 \tag{7}$$

c) Given the Energy-momentum tensor

$$T^{\mu\nu}_{em} = F^{\mu}_{\ \rho} F^{\rho\nu} + \frac{1}{4}\, g^{\mu\nu}\, F_{\rho\sigma} F^{\rho\sigma} \tag{8}$$

compute the components T^{00}_{em}, T^{0i}_{em}, T^{ij}_{em} in terms of the $\vec{E}$ and $\vec{B}$ fields.

d) Show that the Levi-Civita tensor $\epsilon^{\mu\nu\rho\sigma}$ is invariant under Lorentz transformations.