Exercise 1. ***d'Alembert operator and reference frames***

The d'Alembert operator is defined as

$$\Box = \frac{1}{c^2}\frac{\partial}{\partial t^2} - \vec{\nabla}^2. \tag{1}$$

Is this operator invariant under a change of reference frame? For this, check how the operator transforms under coordinate transformations. Consider relative motion in x direction under

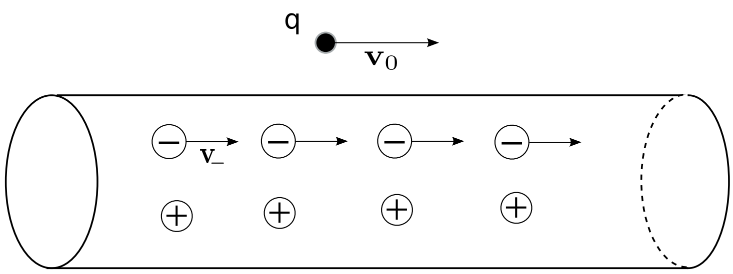
a) Galilean transformations

b) Special Relativity

What is the significance of this in the context of Maxwell's equations?

Exercise 2. ***Electric and magnetic fields in different frames***

Consider a wire and let the laboratory frame $\mathcal{S}$ be the one where the wire is at rest, with a charged particle (with charge q) moving with constant speed $\mathbf{v}_0$ parallel to the wire at a distance r, as in the figure below. The radius of the wire is small in comparison with r.



Consider a current flowing through the wire as the flow of negative charges having all the same velocity $\mathbf{v}_-$. Moreover, let both the positive and negative charge densities ρ_+, ρ_- be constant and uniform inside the wire, and let the wire be globally neutral in the frame $\mathcal{S}$.

a) Evaluate the force $\mathbf{F}$ acting on the charged particle in the laboratory frame $\mathcal{S}$.

b) Evaluate the charge densities ρ_+, ρ_- in the rest frame $\mathcal{S}'$ of the particle. Show that in this frame the net charge density is nonzero.

Hints:

- If you consider a constant and uniform charge density ρ in its rest frame, then the charge density seen by an observer moving with speed $\mathbf{u}$ with respect to the charges is

$$\rho'|_{\text{boost}} = \frac{\rho|_{\text{rest}}}{\sqrt{1-\frac{u^2}{c^2}}};$$

- Remember the proper formula for adding velocities in Special relativity.

c) Evaluate the force $\mathbf{F}'$ acting on the particle in the frame $\mathcal{S}'$ and show that

$$|\mathbf{F}'| = \frac{|\mathbf{F}|}{\sqrt{1 - \frac{v_0^2}{c^2}}}.$$

d) Now consider another frame $\mathcal{S}''$ which moves at a velocity $v_{S''}$ with respect to the laboratory frame $\mathcal{S}$. What is the magnitude of the force, $|\mathbf{F}''|$ acting on the charged particle in this frame?

Hints: Now the charged particle is not at rest, and also the net charge in the wire is not zero. Therefore both electric and magnetic fields contribute to the total electromagnetic force. So you need to rewrite the magnetic field that you found in part a in this boosted frame. Take also into account the fact that the positive charges move in this frame. To find the force due to the electric field recycle your results from parts b and c by just replacing the velocity of the frame $\mathcal{S}'$ by the velocity of the frame $\mathcal{S}''$.

Exercise 3. *Green's functions*

The Green's function physically represents a response of the system if a unit point source (charge) is applied to the system. Mathematically, the Green's function is the kernel of an integral operator that represents an inverse of the differential operator.

Consider the following problem:

$$Au(\vec{x}) = f(\vec{x}) \tag{2}$$

$$B_1 u = \vec{a} \tag{3}$$

$$B_2 u = \vec{b} \tag{4}$$

Where A is a differential operator (e.g. Laplacian), and B_1 and B_2 are boundary value operators (compare the discretised version of this with Sheet 3, question 3). We can write the differential operator A together with boundary conditions B, as a differential operator L, s.t:

$$Lu(x) = f(x) \tag{5}$$

1. When does the inverse of the L exist?
2. If the inverse exists, what form does it take? Can you recognise the Green's function?
3. If f represents a unit point source acting at the point y, what does the Green's function represent?
4. Compare the discretised version of taking the inverse of operator L, with the matrix inversion in solving the matrix equations (e.g. solving Poisson equation numerically in Sheet 3, Question 3). What is the important difference between the problem in Eq.5 and the matrix equations?

Exercise 4. *Getting more familiar with magnetostatics*

In this exercise, we would like to understand a few more things about magnetic field and vector potential. To this end, we will first consider a sheet with current and then ask for more general considerations in the context of boundary conditions in magnetostatics.

a) Imagine a sheet with surface current $\vec{K}$. What can you say about the magnetic field above and below the surface? How does this compare to the electric field in the presence of surface charges?

b) Take a sphere of radius R. Suppose that we know the vector potential on the surface is given by $\vec{A}_S$, how can we calculate the potential outside the sphere? How did we do it in electrostatics? Give an outline only!