Exercise 1. *Electrostatic potential from a dipole*

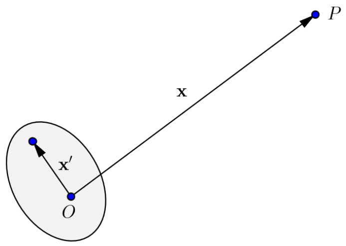


Figure 1: Potential at large distance from the charge distribution

1. The general expression for the potential is given by,

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'. \tag{1}$$

Show that at large distances from the charge distribution ($|\mathbf{x}| \gg |\mathbf{x}'|$), as it is shown in figure 1, the potential can be approximated by

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\Big[\frac{q}{r} + \frac{\mathbf{p}\cdot\mathbf{x}}{r^3} + \dots\Big]. \tag{2}$$

Where $r = |\mathbf{x}|$, q is the total charge and $\mathbf{p}$ is the electric dipole moment, defined by

$$\mathbf{p} = \int \mathbf{x}'\rho(\mathbf{x}')d^3x' \tag{3}$$

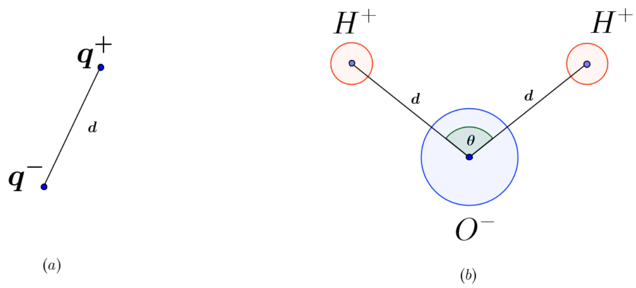


Figure 2: Different examples of dipoles

2. Calculate the potential of the following configurations (see figure 2 for (a) and (b) and figure 3 for (c).) by applying (2),

   (a) Two identical point charges with opposite sign at distance d.

   (b) The potential given by a molecule of water ($d_{OH} = 0,96.10^{-10}m$, $\theta = 104,5$, $Q_{O^-} = -0,66\,e$ and $Q_{H^+} = 0,33\,e$)

   (c) Four charges forming two dipoles, according to figure 3. For which angles θ_1 and θ_2 and distances $d_1 = |\mathbf{d}_1|$ and $d_2 = |\mathbf{d}_2|$ is the total dipole moment vanishing? How are the dipoles disposed in that case?

Exercise 2. ***Van der Waals forces between 2 dipoles***

In this exercise we will see how the van der Waals forces between two molecules or atoms emerge. To this end, consider the situation depicted in the Figure 3.

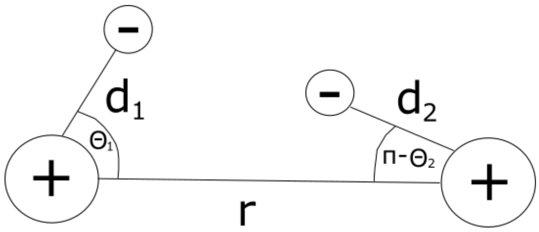


Figure 3: The system consisting of 2 dipoles.

Let $\boldsymbol{r}$ be the vector connecting the 2 positive charges (of the charge $+q$ each), and $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ the vectors connecting each positive charge with a negative charge ($-q$) it forms a dipole with.

1. To start with, write down the expression for the total energy of the system. In total you should get the contribution from 6 terms. Make sure that each of them comes with a correct sign.

2. Now prove that for $|\boldsymbol{r}| >> |\boldsymbol{a}|$ we have the following expansion:

$$\frac{1}{|\boldsymbol{r}+\boldsymbol{a}|} = \frac{1}{|\boldsymbol{r}|}\left(1 - \frac{\hat{r}\cdot\boldsymbol{a}}{|\boldsymbol{r}|}\right) + \frac{1}{2|\boldsymbol{r}|^3}\left(3(\hat{r}\cdot\boldsymbol{a})^2 - |\boldsymbol{a}|^2\right) + O\left(\left(\frac{|\boldsymbol{a}|}{|\boldsymbol{r}|}\right)^3\right) \tag{4}$$

3. Use the result of the previous point to simplify the expression for the energy of the configuration you obtained in the first part of the exercise. To this end, assume that $|\boldsymbol{r}| >> |\boldsymbol{d}_1|,\ |\boldsymbol{d}_2|$.

4. In the new expression you have just obtained identify the terms that are due to the interactions between two dipoles. We will call their sum the *interaction energy* U_{int}.

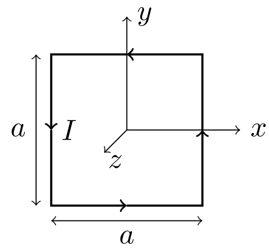
5. Now express the scalar products between the vectors $\boldsymbol{r},\ \boldsymbol{d}_1,\ \boldsymbol{d}_2$ using trigonometric functions in order to obtain for the U_{int} the following expression:

$$U_{int} = \frac{q^2}{4\pi\epsilon_0}\frac{|\boldsymbol{d}_1||\boldsymbol{d}_2|}{|\boldsymbol{r}|^3}\left(\sin\theta_1\sin\theta_2 - 2\cos\theta_1\cos\theta_2\right) \tag{5}$$

6. The physical systems tend to minimize the interaction energy. Find the values of θ_1 and θ_2 such, that U_{int} is at it's lowest.
7. Finally calculate the force $\boldsymbol{F}$ with which the dipoles act at each other (assume that the angles $\theta_{1,2}$ already satisfy the minimizing condition you found in the previous part) by the means of the formula $\boldsymbol{F} = -\boldsymbol{\nabla} U_{int}$. This is a good approximation as long as the distance between the dipoles is large enough. What further corrections should we include if the dipoles come closer to each other?

Exercise 3. ***Magnetic dipole***

Consider a square current loop in the xy-plane of side a. The center of the square is placed in the origin and the sides are parallel to the coordinate-axes.



1. Show that for $r \gg a$ the vector potential $\vec{A}$ is
$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{\mu} \times \vec{r}}{r^3}, \tag{6}$$
where $\vec{\mu} = a^2 I \vec{e}_z$ and I is the intensity of the current that flows in the loop.
Hint: $\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int_\gamma \frac{d\vec{r}\,'}{|\vec{r} - \vec{r}\,'|}$.
2. Compute the magnetic field $\vec{B}$ for $r \gg a$ and compare it to the electric field $\vec{E}$ of an electric dipole.

Exercise 4. ***The Cosmic Microwave Background (CMB) Sky***

The theoretical and experimental CMB power spectra are customarily presented in the context of spherical harmonic multipoles. In the following link

`http://find.spa.umn.edu/~pryke/logbook/20000922/`

you can find examples of multipoles plots and an application to the CMB Sky. Read it carefully and try to understand the link between harmonic multipoles and one of their application in cosmology.