Exercise 1. *Vector Identities.*

In Electrodynamics we frequently use standard vector identities. To practice with Einstein summation convention prove the following identities:

1. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$
2. $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})\,(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})\,(\mathbf{b} \cdot \mathbf{c})$
3. $\mathbf{Ra} \times \mathbf{Rb} = \mathbf{R}\,(\mathbf{a} \times \mathbf{b})$
4. $\nabla \times \nabla\psi = 0$
5. $\nabla \cdot (\nabla \times \mathbf{A}) = 0$
6. $\nabla \times (\nabla \times \mathbf{A}) = \nabla\,(\nabla \cdot \mathbf{A}) - \Delta\mathbf{A}$
7. $\nabla\,(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\,\mathbf{B} + (\mathbf{B} \cdot \nabla)\,\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are vectors, $\mathbf{A}$, $\mathbf{B}$ are vectorfields, ψ is a function and $\mathbf{R} \in \mathrm{SO}(3)$. Moreover assume that all components A_i, B_j and also ψ are in $\mathcal{C}(2)$, i.e. two times continuously differentiable.
Don't write out cross products explicitly, but use the index notation involving the Levi-Civita symbol ε_{ijk}.

Exercise 2. *Gauss and Stokes theorems.*

1. Consider the vector field in $\mathbb{R}^3$ (in Cartesian coordinates)

$$\mathbf{V}(x, y, z) = (x\,y,\; z^2\,y^2,\; z^2 + y), \tag{1}$$

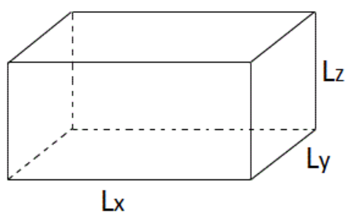
and a parallelepiped domain

$$\mathcal{D} = \{(x, y, z) \in \mathbb{R}^3 \,|\, 0 \le x \le L_x,\, 0 \le y \le L_y,\, 0 \le z \le L_z, \}. \tag{2}$$

Check the validity of the divergence theorem, by proving that

$$\int_{\mathcal{D}} \mathrm{d}^3x\, \nabla \cdot \mathbf{V} \;=\; \int_{\partial\mathcal{D}} \mathbf{V} \cdot \mathrm{d}\hat{\mathbf{A}}, \tag{3}$$

where $\partial\mathcal{D}$ is the border surface of the parallelepiped $\mathcal{D}$ in figure.



Note: Given a surface $\mathbf{A}$, parametrized as $\mathbf{A} = \{A_x(s,t), A_y(s,t), A_z(s,t)\}$, the surface vector $\mathrm{d}\hat{\mathbf{A}}$ is defined as

$$\mathrm{d}\hat{\mathbf{A}} = \frac{\partial \mathbf{A}}{\partial s} \times \frac{\partial \mathbf{A}}{\partial t}. \tag{4}$$

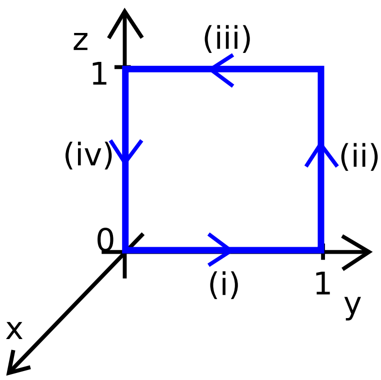
When parametrizing the parallelepiped, pay attention to the orientation of the surfaces.

2. Consider the vector field

$$\mathbf{V}(x,y,z) = \left(0, 6xz + 9y^2, 12yz^2\right) .$$

Check that $\mathbf{V}$ fulfills Stokes' theorem for the area/path defined in the figure, i.e. calculate both sides of the equation

$$\oint \mathbf{V} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{V}) \cdot d\hat{\mathbf{A}} . \tag{5}$$

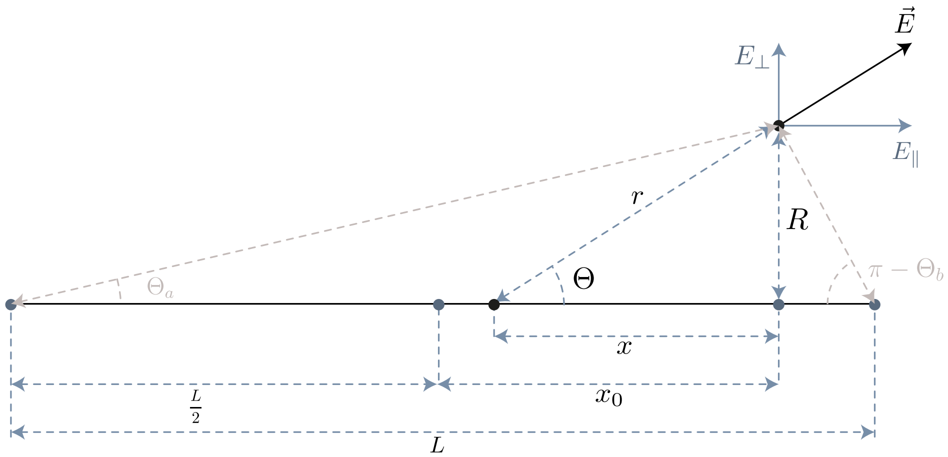


Exercise 3. ***Electric field from a charged line.***

1. Consider an infinite line with constant charge density $\lambda = q/L$.
   (a) Using Gauss law, find the value of the electric field $\vec{E}$ generated by the line.
   (b) Compute $\vec{E}$ again, now using Coulomb's law.
   Hint. Find first the components of the electric field parallel ($E_{\parallel}$) and perpendicular ($E_{\perp}$) to the line, as described in the picture.
2. Consider now a charged line of length L.
   (a) Compute the two components $E_{\parallel}$ and $E_{\perp}$ of the electric field.
   Hint. Introduce a parameter x_0 related to a shift from the middle of the line. Be careful with the integration limits!
   (b) Take the limit for $L \to \infty$. You should recover the same values as in (1b).
3. Explore the limit $L/2 \ll (x_0, R)$. In this case the observation point is very far and we expect to recover the $1/r^2$ behavior of a point-like charge.

   Is it enough to expand up to first order, or do you need one more?

   Hint. Notice that now x_0 is not part of the line anymore, but is somewhere very far from it. Therefore the definition of the two angles θ_a and θ_b needs to be changed. Also, in order to compare the result with your expectation, write it in terms of radial and tangential components with respect to a spherical surface.


$\vec{E}$
$E_\perp$
$E_\parallel$
r
R
Θ
Θ_a
$\pi - \Theta_b$
x
$\frac{L}{2}$
x_0
L