Exercise 1. Reminders: Covariant Formalism / Static vs. Dynamical Equations

Goal: You may choose to revise some aspects of the covariant formalism in Minkowski space, and/or Greens functions in static versus dynamical equations of electrodynamics.

1.1. *Covariant Formalism.* The four-dimensional Minkowski space carries the metric $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. Consider a Lorentz transformation of the coordinates

$$x^\mu \to \bar{x}^\mu = \Lambda^\mu{}_\nu \, x^\nu \; . \tag{1}$$

(a) How do the following objects transform?

(i) $\partial/\partial x^\mu$, the gradient operator;

(ii) $g_{\mu\nu}$, the metric itself;

(iii) The four-vector $A^\mu = (\phi, \vec{A})$ can be shown to transform as $\bar{A}^\mu = \Lambda^\mu{}_\nu A^\nu$. Assuming this, how does $A_\mu := (\phi, -\vec{A})$ transform?

(iv) $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the covariant electromagnetic field tensor.

Which of the above are covariant or contravariant vectors?

(b) Given an object V^μ which transforms as a contravariant vector, how can we construct an object V_μ which transforms like a covariant vector?

1.2. *Green's Functions, Static and Dynamical Equations.* Consider the differential equations for ϕ and $\vec{A}$:

$$\nabla^2 \phi\,(\vec{r}, t) = -4\pi \rho\,(\vec{r}, t) \;\; ; \tag{2a}$$

$$\nabla^2 \vec{A}\,(\vec{r}, t) - \frac{1}{c^2}\partial_t^2 \vec{A}\,(\vec{r}, t) = -\frac{4\pi}{c}\vec{j}_\perp\,(\vec{r}, t) \;\; . \tag{2b}$$

Calculate the Green's function for each equation. Why do we say that the first equation is a "static" differential equation, and that the second one is "dynamical"?

Exercise 2. Energie des Quantisierten Elektromagnetischen Feldes

Lernziel: Wir gehen hier die Rechnung des Hamiltonoperator des Strahlungsfeldes wieder durch. Damit üben wir auch Manipulationen mit den Feldoperatoren.

Durch die Quantisierung des Strahlungsfeldes (ohne Quellen) kann das Vektorpotential neu als Operator geschrieben werden. Man bezeichnet es dann als Feldoperator, der gegeben ist durch

$$\hat{\boldsymbol{A}}(\boldsymbol{r}, t) = \frac{1}{\sqrt{L^3}} \sum_{\boldsymbol{k},\lambda} \sqrt{\frac{2\pi\hbar c^2}{\omega_{\boldsymbol{k}}}} \Big(\hat{a}_{\boldsymbol{k},\lambda} \boldsymbol{\varepsilon}(\boldsymbol{k}, \lambda) e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega_{\boldsymbol{k}} t)} + \hat{a}^\dagger_{\boldsymbol{k},\lambda} \boldsymbol{\varepsilon}^*(\boldsymbol{k}, \lambda) e^{-i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega_{\boldsymbol{k}} t)} \Big). \tag{3}$$

Dabei sind $\hat{a}_{\boldsymbol{k},\lambda}$ und $\hat{a}^\dagger_{\boldsymbol{k},\lambda}$ die Auf- und Absteigeoperatoren der Moden des Strahlungsfeldes und für die Dispersionsrelation gilt $\omega_{\boldsymbol{k}} = c\,|\boldsymbol{k}|$. Die Polarisationsvektoren sind orthonormal

$\boldsymbol{\varepsilon}^*(\boldsymbol{k},\lambda)\cdot\boldsymbol{\varepsilon}(\boldsymbol{k},\lambda') = \delta_{\lambda\lambda'}$ und in der Coulomb-Eichung ist die Polarisation transversal $\boldsymbol{\varepsilon}(\boldsymbol{k},\lambda)\cdot\boldsymbol{k} = 0$. Das elektrische und das magnetische Feld sind gegeben durch $\hat{\boldsymbol{E}} = -\frac{1}{c}\frac{\partial}{\partial t}\hat{\boldsymbol{A}}$ und $\hat{\boldsymbol{B}} = \nabla \times \hat{\boldsymbol{A}}$.

Zeige, dass die Energie des Strahlungsfeldes ist gegeben durch

$$\hat{H}_{SF} = \frac{1}{8\pi}\int_V d^3x \left(|\hat{\boldsymbol{E}}|^2 + |\hat{\boldsymbol{B}}|^2\right) = \sum_{\boldsymbol{k},\lambda} \hbar\omega_{\boldsymbol{k}} \left(\hat{a}^\dagger_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda} + \frac{1}{2}\right) . \tag{4}$$

Exercise 3. Photon Spin and Helicity

Goal: The angular momentum of a particle tells us how the quantum state transforms under rotations of real space. The vector nature of the field operator gives an intrinsic transformation under rotations which corresponds to spin. Here we convince ourselves that the photon has helicity ± 1, corresponding to spin states $m = \pm 1$.

Consider a single photon with definite momentum $\boldsymbol{k}$, in any linear combination of the two possible polarization states

$$|\psi\rangle = c_1\, a^\dagger_{\boldsymbol{k},1}|0\rangle + c_2\, a^\dagger_{\boldsymbol{k},2}|0\rangle \ . \tag{5}$$

To study the spin of this photon, we will consider an infinitesimal rotation about the axis $\boldsymbol{k}$ and of angle $\delta\phi$.

(a) Introduce the so-called *circular polarization vectors*

$$\boldsymbol{\varepsilon}^\pm = \mp\frac{1}{\sqrt{2}}\left(\boldsymbol{\varepsilon}^{(1)} \pm i\boldsymbol{\varepsilon}^{(2)}\right) \ , \tag{6}$$

and show that under the infinitesimal rotation described above, we have

$$\delta\boldsymbol{\varepsilon}^\pm = \mp i\,\delta\phi\,\boldsymbol{\varepsilon}^\pm \ . \tag{7}$$

(b) Why are the $\boldsymbol{\varepsilon}^\pm$ called *circular* polarization vectors? Justify this terminology by considering classical fields.

(c) How does $|\psi\rangle$ transform under this infinitesimal rotation? Identify the generator $S_{\boldsymbol{k}}$ of this rotation of $|\psi\rangle$, and argue why $S_{\boldsymbol{k}}$ may be interpreted as the spin operator in the direction $\boldsymbol{k}$. Which values of m may the photon have?

Hint: In principle, we should have considered a general rotation of the field operator about any axis; however the only type of rotation relevant for the spin is in the plane of the two allowed polarization vectors.

(d) Deduce the possible values for the helicity of the photon with definite momentum $\boldsymbol{k}$, where the helicity operator is defined as $h = \boldsymbol{S}\cdot\frac{\boldsymbol{k}}{|\boldsymbol{k}|}$.